Tutorial on Robust Auction Design
Lecture 1

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IPV auction design problem (Myerson, 1981)

- N buyers, one seller
- A single unit of a good for sale
- The buyers have **independent and private values** (IPV)
- $v_i \sim F_i \in \Delta(V_i)$, with positive density f_i, and $V_i = [0, \bar{v}]$
- We let $f(v)$ denote the joint density of $(v_1, \ldots, v_N)$
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- The outcome consists of **allocations** $q \in \mathbb{R}_+^N$ satisfying $\sum_i q_i \leq 1$ and **transfers** $t \in \mathbb{R}^N$
- Agent's have quasilinear preferences over probabilities of receiving the good and transfers (to the seller): for $i \geq 1$,

$$u_i(v_i, q, t) = v_i q_i - t_i$$

- Seller gets $u_0(q, t) = \sum_i t_i$, i.e., wants to maximize revenue.

Auction mechanisms

- A **(auction) mechanism** $\mathcal{M}$ consists of
  (i) A measurable set of actions A_i that player i can take;
  (ii) A pair of measurable mappings

$$q : A \to \mathbb{R}^N_+, \text{ st } \sum_i q_i(a) \leq 1$$

$$t : A \to \mathbb{R}^N$$

  where $A = \times_{i=1}^N A_i$.

Strategies and equilibrium

- Strategies and Bayes Nash equilibria are defined as usual
- A mechanism $\mathcal{M}$ induces a Bayesian game among the buyers
- A strategy for player i is a measurable mapping $b_i : V_i \to \Delta(A_i)$
- Under the strategy profile b, v_i's interim expected payoff is

$$U_i(b; v_i, \mathcal{M}) = \int_{v_{-i} \in [0,\overline{v}]^{n-1}} \int_{a \in A} u_i(v_i, q(a), t(a)) b(da \mid v_i, v_{-i}) f_{-i}(v_{-i}) dv_{-i}$$

- A profile of strategies is a **Bayes Nash equilibrium** (BNE) if $U_i(b; \mathcal{M}) \geq U_i(b_i', b_{-i}; \mathcal{M})$ for all i, b_i'

The seller's problem

- We will assume that players can always "opt out" of the mechanism and obtain a payoff from zero, even after they know their values
- Thus, the a mechanism and equilibrium will be played only if that are **individually rational** (IR), meaning that

$$\int_{v_{-i} \in [0,\overline{v}]^{n-1}} \int_{a \in A} (v_i q_i(a) - t_i(a))\, b(da \mid v_i, v_{-i}) f_{-i}(v_{-i}) dv_{-i} \geq 0$$

- The seller's problem is to maximize expected revenue, i.e,

$$\Pi(b; \mathcal{M}) = \sum_{i=1}^{n} \int_{v \in [0,\overline{v}]^n} \int_{a \in A} t_i(a) b(da \mid v) f(v) dv$$

  over all mechanisms $\mathcal{M}$ and BNE b subject to IR
- An **optimal auction** is a mechanism that solves the seller's problem

The revelation principle

- *Without loss to use* **direct mechanisms***, in which* $A_i = V_i$*, and take* $b_i(\{v_i\} \mid v_i) = 1$ *as the BNE*

- Suppose b is a BNE of the mechanism $\mathcal{M} = (\{A_i\}, q, t)$
- Players report their values to $\mathcal{M}' = (\{V_i\}, q', t')$, which "simulates" b for them:

$$q'(v) = \int_{a \in A} q(a) b(da \mid v), \quad t'(v) = \int_{a \in A} t(a) b(da \mid v)$$

- The equilibrium strategy in $\mathcal{M}'$ is just $b_i'(\{v_i\}|v_i) = 1$

Incentive compatibility for direct mechanisms

- We say that a direct mechanism is **incentive compatible** (IC) if reporting your true value is an equilibrium
- Let $Q_i(v_i)$ and $T_i(v_i)$ denote the expected allocation and transfers under an incentive compatible direct mechanism:

$$Q_i(v_i) = \int_{v_{-i} \in V_{-i}} q_i(v_i, v_{-i}) f_{-i}(v_{-i}) dv_{-i}$$

$$T_i(v_i) = \int_{v_{-i} \in V_{-i}} t_i(v_i, v_{-i}) f_{-i}(v_{-i}) dv_{-i}$$

- Then v_i's expected payoff if he reports w_i is just $v_i Q_i(w_i) - T_i(w_i)$

Monotonic allocation

Lemma

$\mathcal{M}$ is IC if and only if Q_i is increasing for every i.

Only if:

- If $v_i > v_i'$, then

$$v_i Q_i(v_i) - T_i(v_i) \geq v_i Q_i(v_i') - T_i(v_i')$$
$$v_i' Q_i(v_i') - T_i(v_i') \geq v_i' Q_i(v_i) - T_i(v_i)$$

- Adding these together, we get

$$(v_i - v_i')(Q_i(v_i) - Q_i(v_i')) \geq 0$$

- Thus, $Q_i(v_i) \geq Q_i(v_i')$

If: Verify using the transfer formula from the next slide

The envelope formula

- Note that type v_i's surplus in equilibrium is

$$U_i(v_i) = v_i Q_i(v_i) - T_i(v_i) = \max_{w_i} v_i Q_i(w_i) - T_i(w_i)$$

- Can use monotonicity of Q_i to show that U_i is continuous and a.e. differentiable, and the envelope formula holds, i.e.,

$$\frac{d}{dv_i} U_i(v_i) = Q_i(v_i)$$

- Thus,

$$U_i(v_i) = U_i(0) + \int_{x=0}^{v_i} Q_i(x) dx$$

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$$U_i(v_i) = U_i(0) + \int_{x=0}^{v_i} Q_i(x)dx$$

$$T_i(v_i) = v_i Q_i(v_i) - U_i(v_i) = v_i Q_i(v_i) - \int_{x=0}^{v_i} Q_i(x)dx - U_i(0)$$

Virtual value

- Since U_i is increasing, IR is equivalent to $U_i(0) \geq 0$, and obviously revenue is maximized by setting $U_i(0) = 0$
- The seller's revenue is therefore

$$\Pi = \sum_{i=1}^{N} \int_{v_i \in V_i} \left(v_i Q_i(v_i) - \int_{x=0}^{v_i} Q_i(x)dx \right) f_i(v_i)dv_i$$
$$= \sum_{i=1}^{N} \int_{v \in V} \underbrace{\left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right)}_{=\phi_i(v_i)} q_i(v)f(v)dv$$

- Call $\phi_i(v_i)$ **virtual value**
- Difference between v_i and $\phi_i(v_i)$ is the "information rent" collected by type v_i.
- **Regular case**: ϕ_i is increasing for every i

The optimal auction

Theorem

In the regular case, the mechanism with allocation

$$q_i^*(v) = \begin{cases} \frac{1}{|\arg\max_j \phi_j(v_j)|} & \phi_i(v_i) \geq \phi_j(v_j)\,\forall j, \text{ and } \phi_i(v_i) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and transfer given by the envelope formula maximizes the seller's expected revenue.

- Since $q_i^*(v_i, v_{-i})$ is increasing in v_i, truth-telling is the dominant strategy in the optimal mechanism.
- Symmetric bidders: Second price auction with reserve price $\phi_i^{-1}(0)$ is optimal.
  - First-price auction with reserve price $\phi_i^{-1}(0)$ is also optimal.

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A dual perspective on the optimal revenue

- Suppose $V_i = \{v^0, v^1, v^2, \dots, v^M\}$, where $v^0 = 0$, $v^M = \bar{v}$, $v^m - v^{m-1} = \gamma > 0$ for every m
- Consider the Lagrangian:

$$
\begin{aligned}
\mathcal{L} = & \sum_{i,v} f(v) t_i(v) \\
& + \sum_{i,v_i} \alpha_i(v_i) \sum_{v_{-i}} [v_i(q_i(v) - q_i(v_i - \gamma, v_{-i})\mathbb{I}_{v_i > 0}) - (t_i(v) - t_i(v_i - \gamma, v_{-i})\mathbb{I}_{v_i > 0})] \\
& \qquad \cdot f_{-i}(v_{-i})
\end{aligned}
$$

- $\alpha_i(v_i)$ is the multiplier on local downward IC constraint if $v_i > 0$, and on IR constraint if $v_i = 0$

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- $\alpha_i(v_i)$ is the multiplier on local downward IC constraint if $v_i > 0$, and on IR constraint if $v_i = 0$
- Since $t_i(v)$ is a free variable, for $\mathcal{L}$ to be bounded we must have

$$
f_i(v_i) + \alpha_i(v_i) - \alpha_i(v_i + \gamma) \mathbb{I}_{v_i < \bar{v}} = 0,
$$

i.e., $\alpha_i(v_i) = \sum_{v_i' \geq v_i} f_i(v_i')$.

A dual perspective on the optimal revenue

- Substituting $\alpha_i(v_i) = \sum_{v_i' \geq v_i} f_i(v_i')$ into $\mathcal{L}$ gives:

$$
\begin{aligned}
\mathcal{L} &= \sum_{v,i} \alpha_i(v_i) v_i [q_i(v) - q_i(v_i - \gamma, v_{-i}) \mathbb{I}_{v_i > 0}] f_{-i}(v_{-i}) \\
&= -\sum_{v,i} [\alpha_i(v_i + \gamma)(v_i + \gamma) - \alpha_i(v_i) v_i] q_i(v) f_{-i}(v_{-i})
\end{aligned}
$$

A dual perspective on the optimal revenue

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- We get the optimal revenue with *discrete virtual value*:

$$\begin{aligned}\mathcal{L} &= \sum_{v,i} [v_i f_i(v_i) - \gamma \alpha_i(v_i + \gamma)] q_i(v) f_{-i}(v_{-i}) \\ &= \sum_{v,i} \left[v_i - \frac{\sum_{v_i' > v_i} f_i(v_i')}{f_i(v_i)/\gamma} \right] q_i(v) f(v)\end{aligned}$$

Interdependent values (Bulow and Klemperer, 1996)

- Suppose $v_i(s_i, s_{-i})$, where $s_i \sim F_i$, independently distributed
- s_i is bidder i's type or signal
- Virtual value:

$$\phi_i(s) = v_i(s) - \frac{1 - F_i(s_i)}{f_i(s_i)} \cdot \frac{\partial v_i(s)}{\partial s_i}$$

- Suppose $\phi_i(s)$ is increasing in s_i

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- Suppose $\phi_i(s)$ is increasing in s_i
- **The optimal mechanism allocates the good to the bidder with the highest virtual value, as long as it is positive.**

A model with correlated private values

- Follows Crémer and McLean (1988)
- Each bidder has finite set of types S_i, $S = \times_{i=1}^{N} S_i$
- There is a valuation function $v_i : S_i \to \mathbb{R}$
- Common prior $\pi \in \Delta(S)$, which induces conditional distributions $\pi(s_{-i} \mid s_i)$

Mechanisms

- The revelation principle continues to hold, so it is WLOG to restrict attention to direct mechanisms, i.e.,

$$q : S \to \mathbb{R}_+^N, \sum_i q_i(s) \leq 1, \quad t : S \to \mathbb{R}^n$$

- The mechanism is **incentive compatible** (IC) if for all i, s_i, and s_{-i},

$$\sum_{s_{-i}} \pi(s_{-i} \mid s_i) \left(v_i(s_i) q_i(s_i, s_{-i}) - t_i(s_i, s_{-i}) \right)$$
$$\geq \sum_{s_{-i}} \pi(s_{-i} \mid s_i) \left(v_i(s_i) q_i(s_i', s_{-i}) - t_i(s_i', s_{-i}) \right)$$

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- The mechanism is **individually rational** (IR) if for all i and s_i,

$$\sum_{s_{-i}} \pi(s_{-i} \mid s_i) \left(v_i(s_i) q_i(s_i, s_{-i}) - t_i(s_i, s_{-i}) \right) \geq 0$$

Towards full surplus extraction

- Let TS denote the efficient surplus

$$TS = \sum_{s \in S} \pi(s) \max_{i=1,\ldots,n} v_i(s_i)$$

- Given enough linear independence in interim beliefs/correlation in values, **there exist IC and IR mechanisms such that revenue is equal to TS**

- The basic strategy is as follows:
  - Start with a second-price auction to efficiently allocate the good
  - Extract agents' rents from the SPA using side bets

Full surplus extraction

Theorem (Crémer and McLean)

Suppose that for all i and s_i, there **do not** *exist $\{\rho(s_i') \geq 0\}_{s_i' \neq s_i}$ such that*

$$\pi(s_{-i} \mid s_i) = \sum_{s_i' \neq s_i} \rho(s_i')\pi(s_{-i} \mid s_i')$$

for all $s_{-i} \in S_{-i}$. Then, there exists an IC and IR mechanism whose revenue is TS.

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for all $s_{-i} \in S_{-i}$. *Then, there exists an IC and IR mechanism whose revenue is* TS.

- Proof: The allocation is defined by

$$W(s) = \{i : v_i(s) = \max_{j=1,\ldots,n} v_j(s_j)\}$$

$$q_i(s) = \frac{1}{|W(s)|} \mathbb{I}_{i \in W(s)}$$

  i.e., $q_i(s)$ randomizes the allocation among the bidders with high values

- Now, we will construct transfers such that the IC constraints are satisfied and IR is satisfied as an **equality** for all i

Proof, continued

- The hypothesis of the theorem implies that $\pi(s_{-i} \mid s_i)$ (viewed as an element of $\mathbb{R}^{S_{-i}}$) is not in the convex cone generated by

$$\{\pi(s_{-i} \mid s_i') : s_i' \in S_i \setminus \{s_i\}\}$$

Proof, continued

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- By Farkas/SHT, there exists a separating hyperplane $g_i(s_i) \in \mathbb{R}^{S_{-i}}$ such that

$$\sum_{s_{-i} \in S_{-i}} g_i(s_{-i} \mid s_i)\pi(s_{-i} \mid s_i) = 0$$

$$\sum_{s_{-i} \in S_{-i}} g_i(s_{-i} \mid s_i)\pi(s_{-i} \mid s_i') > 0 \ \forall s_i' \neq s_i$$

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- We then set the transfers to be

$$t_i(s) = q_i(s) v_i(s_i) + \kappa g_i(s_{-i} \mid s_i)$$

for some large κ

Proof, continued continued

- Now, observe that the transfer is

$$\sum_{s_{-i} \in S_{-i}} \pi(s_{-i} \mid s_i) q_i(s_i, s_{-i}) v_i(s_i)$$

if the player tells the truth, so that equilibrium surplus is zero

Proof, continued continued

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$$\sum_{s_{-i} \in S_{-i}} \pi(s_{-i} \mid s_i) q_i(s_i, s_{-i}) v_i(s_i)$$

  if the player tells the truth, so that equilibrium surplus is zero
- The transfer from misreporting s_i' is

$$\sum_{s_{-i} \in S_{-i}} \pi(s_{-i} \mid s_i) q_i(s_i', s_{-i}) v_i(s_i) + \kappa \sum_{s_{-i} \in S_{-i}} \pi(s_{-i} \mid s_i) g_i(s_{-i} \mid s_i')$$

- Pick a sufficiently big κ. $\square$

Common value and correlated signals

- Suppose all buyers have the same, ex post value v
- Conditional on v, buyers receives iid signals s_i (bidder i only observes s_i)
- "Mineral rights" model

- $v_i(s_1, \ldots, s_N) = \mathbb{E}[v \mid (s_1, \ldots, s_N)]$
- Since v is not observed, $(s_1, \ldots, s_N)$ is correlated.

- Adapt Crémer-McLean FSE when S is finite:
  - Start with any full allocation of the good
  - Construct side bets as before
- See McAfee, McMillan, and Reny (1989) and McAfee and Reny (1992) for FSE under infinite signals

Critique of full surplus extraction

- If the matrices $\{\pi(s_{-i} \mid s_i)\}_{s_i \in S_i}$ are close to singular, then the side bets have to be enormous to deter deviations
  - In other words, very large transfers would be required after certain signal realizations
  - This is problematic if there is limited liability or risk aversion

- Moreover, calibrating these "side bets" requires the seller to have very precise knowledge of beliefs
  - If π is misspecified, the buyers may go from breaking even to losing millions on average (and ditto for the seller)

- Implausibility of FSE motivates work on robust auction that is guaranteed to work well regardless of the belief distribution