Tutorial on Robust Auction Design
Lecture 4

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The question

- You are a seller of a good
- You know the set of bidders
- You know the distribution of bidders values
- You are uncertain about the model of bidders' information
- You cannot (or won't) quantify this uncertainty in terms of a Bayesian prior
- What auction should you run?

An answer

- In many cases, the auction you should run is **proportional auction**: $A_i = \mathbb{R}_+$ for each bidder i,

$$q_i^*(a) = \frac{a_i}{\Sigma a} \cdot Q^*(\Sigma a), \qquad t_i^*(m) = \frac{a_i}{\Sigma a} \cdot T^*(\Sigma a),$$

where $\Sigma a = \sum_{i=1}^{N} a_i$,

$$Q^*(\Sigma a) = \begin{cases} \Sigma a / x & \Sigma a < x^*, \\ 1 & \Sigma a \geq x^*. \end{cases}$$

Values

- A single unit for sale
- N bidders
- Value $v_i \in V_i \subset [0, \infty)$, $|V_i| < \infty$
- $v = (v_1, \ldots, v_N)$
- Prior $\mu \in \Delta(V)$

Mechanisms

- A **mechanism** is a triple $\mathcal{M} = (A, q, t)$
  - Finite actions A_i for $i = 1, \ldots, N$
  - Action profiles $A = A_1 \times \cdots \times A_N$
  - Allocations $q : A \to [0,1]^N$, $\Sigma q(a) \leq 1$ ($\Sigma x = x_1 + \cdots + x_N$ for $x \in \mathbb{R}^N$)
  - Transfers: $t : A \to \mathbb{R}^N$

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- **Participation security:** For all i, there exists a action $0 \in A_i$ such that $t_i(0, a_{-i}) = 0 \ \forall v_i, a_{-i}$

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- $\boldsymbol{M}$ set of participation-secure mechanisms

Information structures (aka type spaces)

- An **information structure** is a pair $\mathcal{I} = (S, \sigma)$
  - Finite signals S_i
  - Signal profiles $S = S_1 \times \cdots \times S_N$
  - Joint distribution $\sigma \in \Delta(S \times V)$ where marginal on V is μ
- $\boldsymbol{I}$ is the set of information structures

Equilibrium

- Given $(\mathcal{M}, \mathcal{I})$, (behavioral) strategies $b_i : S_i \to \Delta(A_i)$
- $B(\mathcal{M}, \mathcal{I})$ is the set of **Bayes Nash equilibria**
- Induced profit from b:

$$\Pi(\mathcal{M}, \mathcal{I}, b) = \sum_{v,s,a,i} t_i(a) b(a \mid s) \sigma(s, v)$$

A strong minimax theorem

Theorem

Suppose $\mu(v) > 0$ for all $v \in V$. Then

$$\sup_{\mathcal{M} \in \boldsymbol{M}} \inf_{\mathcal{I} \in \boldsymbol{I}} \inf_{b \in B(\mathcal{M},\mathcal{I})} \Pi(\mathcal{M}, \mathcal{I}, b) = \inf_{\mathcal{I} \in \boldsymbol{I}} \sup_{\mathcal{M} \in \boldsymbol{M}} \sup_{b \in B(\mathcal{M},\mathcal{I})} \Pi(\mathcal{M}, \mathcal{I}, b).$$

- LHS is "MAX-2MIN", RHS is "MIN-2MAX"
- The value of these programs is Π^*, the **profit guarantee**
- Equilibrium selection does not matter!
- $(\mathcal{M}^{\text{MAX-2MIN}}, \mathcal{I}^{\text{MIN-2MAX}})$ is a saddle point
  - Builds on Chung and Ely (2007)

An even stronger theorem

- We construct sequences of linear programs that, for a finite number of actions/signals, bound the MAX-2MIN and MIN-2MAX profits
- For each $k \geq 1$ and i:

$$X_i(k) = \left\{0, \frac{1}{k}, \ldots, \frac{k^2-1}{k}, k\right\}$$

- $X(k) = \times_{i \in N} X_i(k)$

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$$\Pi^{\mathrm{MAX-2MIN}}(k) = \sup_{\mathcal{M} \in \boldsymbol{M}(k)} \inf_{\mathcal{I} \in \boldsymbol{I}} \inf_{b \in B(\mathcal{M}, \mathcal{I})} \Pi(\mathcal{M}, \mathcal{I}, b)$$

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- $\boldsymbol{I}(k)$ is the set of information structures with signal space $X(k)$

$$\Pi^{\mathrm{MIN-2MAX}}(k) = \inf_{\mathcal{I} \in \boldsymbol{I}(k)} \sup_{\mathcal{M} \in \boldsymbol{M}} \sup_{b \in B(\mathcal{M}, \mathcal{I})} \Pi(\mathcal{M}, \mathcal{I}, b)$$

Discrete derivatives

- Let $f : X(k) \to \mathbb{R}^N$, and define the discrete upward partial derivative:

$$\nabla_i^+ f(x) = \mathbb{I}_{x_i < k}(k-1)(f_i(x_i + 1/k, x_{-i}) - f_i(x))$$

- $\nabla^+ f(x) = (\nabla_1^+ f(x), \ldots, \nabla_N^+ f(x))$
- $\nabla^+ \cdot f(x) = \sum_{i=1}^N \nabla_i^+ f(x)$

Linear relaxation for MAX-2MIN

$$
\begin{aligned}
&\underline{\Pi}^{\mathrm{MAX-2MIN}}(k) = \\
&\qquad \max_{\substack{q:X(k)\to\mathbb{R}^N_+,\, t:X(k)\to\mathbb{R}^N, \\ \lambda:V\to\mathbb{R}}} \sum_{v\in V} \mu(v)\lambda(v) \\
&\quad \text{s.t. } \Sigma q(x) \leq 1 \ \forall x; \\
&\qquad t_i(0, x_{-i}) = 0 \ \forall i, x_{-i}; \\
&\qquad \lambda(v) \leq \Sigma t(x) + v \cdot \nabla^+ q(x) - \nabla^+ \cdot t(x) \ \forall v, x
\end{aligned} \tag{1}
$$

- Maximizing a lower bound on revenue across mechanisms, subject to local IC

Linear relaxation for MAX-2MIN

- For a fixed mechanism, minimize the revenue across (local) Bayes correlated equilibria

(BCE)

$$\min_{\sigma \geq 0} \sum_{x,v,i} t_i(x)\sigma(x,v)$$

s.t.

$$\sum_{x} \sigma(x,v) = \mu(v) \ \forall v;$$

$$\sum_{x_{-i},v} \big(v_i \nabla_i^+ q(x_i, x_{-i}) - \nabla_i^+ t(x_i, x_{-i})\big)\sigma(x_i, x_{-i}, v) \leq 0 \ \forall i, x_i$$

(D-BCE)

$$\max_{\alpha \geq 0, \lambda} \sum_{v} \mu(v)\lambda(v)$$

s.t.

$$\lambda(v) \leq \Sigma t(x) + \sum_{i,x_i} \alpha_i(x_i)\big(v_i \nabla_i^+ q(x) - \nabla_i^+ t(x)\big) \ \forall x, v$$

Censored geometric distribution

- Now define

$$\rho_i(x_i) = \left(1 - \frac{1}{k}\right)^{kx_i} \left(\frac{1}{k}\right)^{\mathbb{I}_{x_i < k}}$$

$$\rho(x) = \prod_{i=1}^{N} \rho_i(x_i)$$

- (PMF of the censored geometric with arrival rate $1/k$)

Linear relaxation for MIN-2MAX

$$
\begin{aligned}
&\overline{\Pi}^{\mathrm{MIN-2MAX}}(k) = \\
&\qquad \min_{\substack{\sigma: X(k)\times V \to \mathbb{R}_+,\ w: X(k)\to\mathbb{R}^N_+,\\ \gamma: X(k)\to\mathbb{R}_+}} \quad \sum_{x\in X(k)} \gamma(x) \\
&\qquad \text{s.t.} \quad \sum_{x\in X(k)} \sigma(x,v) = \mu(v)\ \forall v; \\
&\qquad\qquad \sum_{v\in V} \sigma(x,v) = \rho(x)\ \forall x; \\
&\qquad\qquad w(x) = \frac{1}{\rho(x)} \sum_{v\in V} v\sigma(x,v)\ \forall x \\
&\qquad\qquad \gamma(x) \geq \rho(x)\left[w_i(x) - \nabla_i^+ w(x)\right]\ \forall x;
\end{aligned}
\tag{2}
$$

- Minimizing the highest virtual value across information structures where the signal distribution is ρ

Linear relaxations converge as $k \to \infty$

Theorem

For all $k > 0$,

$$\overline{\Pi}^{\mathrm{MIN-2MAX}}(k) \geq \Pi^{\mathrm{MIN-2MAX}}(k) \geq \Pi^{\mathrm{MAX-2MIN}}(k) \geq \underline{\Pi}^{\mathrm{MAX-2MIN}}(k).$$

If $\mu(v) > 0$ *for all* $v \in V$*, then*

$$\lim_{k\to\infty} \overline{\Pi}^{\mathrm{MIN-2MAX}}(k) = \lim_{k\to\infty} \underline{\Pi}^{\mathrm{MAX-2MIN}}(k) = \Pi^*.$$

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Moreover,

- *If (q, t) solves (1), then profit in $(X(k), q, t)$ is at least $\underline{\Pi}^{\mathrm{MAX-2MIN}}(k)$ for any information structure and equilibrium.*
- *If σ solves (2), then profit in $(X(k), \sigma)$ is at most $\overline{\Pi}^{\mathrm{MIN-2MAX}}(k)$ in any mechanism and equilibrium.*

Two programs

(1)

$$\max_{q\geq 0, t, \lambda} \sum_v \mu(v)\lambda(v)$$

$$\begin{aligned}
\text{s.t. } & \Sigma q(x) \leq 1 \ \forall x \\
& t_i(0, x_{-i}) = 0 \ \forall i, x_{-i}; \\
& \lambda(v) \leq \Sigma t(x) \\
& \quad + v \cdot \nabla^+ q(x) - \nabla^+ \cdot t(x) \ \forall x, v
\end{aligned}$$

(2)

$$\min_{\sigma\geq 0, \gamma\geq 0, w} \sum_x \gamma(x)$$

$$\begin{aligned}
\text{s.t. } & \sum_x \sigma(x, v) = \mu(v) \ \forall v; \\
& \sum_v \sigma(x, v) = \rho(x) \ \forall x; \\
& w_i(x) = \frac{1}{\rho(x)} \sum_v v_i \sigma(x, v) \ \forall i, x; \\
& \gamma(x) \geq \rho(x)[w_i(x) - \nabla_i^+ w_i(x)] \ \forall i, x
\end{aligned}$$

Solving out transfers from (1)

- In program (2), we "solved out" the transfers
- Can do the same thing in (1):
  - Let $\Xi(x) = \nabla^+ \cdot t(x) - \Sigma t(x)$ denote the **aggregate excess growth**
  - For fixed Ξ, there exists a t that satisfies this equation iff $\sum_x \rho(x)\Xi(x) = 0$ (implied by, e.g., Farkas' lemma)
- So, in program (1), we can substitute in Ξ for t and add the expectation of Ξ to the objective:

Two programs

$(1')$

$$\max_{\lambda, q \geq 0, t} \sum_v \mu(v)\lambda(v) + \sum_x \rho(x)\Xi(x)$$

$$\text{s.t. } \Sigma q(x) \leq 1 \ \forall x;$$

$$\lambda(v) + \Xi(x) \leq v \cdot \nabla^+ q(x) \ \forall x, v.$$

(2)

$$\min_{\sigma \geq 0, \gamma \geq 0, w} \sum_x \gamma(x)$$

$$\text{s.t. } \sum_x \sigma(x, v) = \mu(v) \ \forall v;$$

$$\sum_v \sigma(x, v) = \rho(x) \ \forall x;$$

$$w_i(x) = \frac{1}{\rho(x)} \sum_v v_i \sigma(x, v) \ \forall i, x;$$

$$\gamma(x) \geq \rho(x)[w_i(x) - \nabla_i^+ w_i(x)] \ \forall i, x$$

(1') and dual of (2)

$(1')$

$$\max_{\lambda,\Xi,q\geq 0} \sum_{v} \mu(v)\lambda(v) + \sum_{x} \rho(x)\Xi(x)$$

$$\text{s.t. } \Sigma q(x) \leq 1 \ \forall x;$$

$$\lambda(v) + \Xi(x) \leq v \cdot \nabla^{+} q(x) \ \forall x, v$$

(D-2)

$$\max_{\Xi,\lambda,q\geq 0} \sum_{v} \mu(v)\lambda(v) + \sum_{x} \rho(x)\Xi(x)$$

$$\text{s.t. } \Sigma q(x) \leq 1 \ \forall x;$$

$$\lambda(v) + \Xi(x) \leq v \cdot \nabla^{-} q(x) \ \forall x, v$$

Shifting

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- We complete the proof of the theorem by showing that (1') and (2) have almost the same value when k is large
- They are "almost" a dual pair, except for the direction of local IC
- Given a feasible q for (D-2), we construct a feasible q' for (1'), so that (1') and (2) have almost the same value
- If q' is non-decreasing, can use:

$$q'_i(x) = \begin{cases} q_i(x_i - 1/k, x_{-i}) & \text{if } x_i > 0; \\ 0 & \text{if } x_i = 0 \end{cases}$$

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- Complication: If q decreases, could have $\Sigma q'(x) > 1$
- Last step: decrease in q is bounded below, and the bound goes to zero faster than $1/k \implies$ we can "rescale" q' to make it feasible without significantly changing the objective

As $k \to \infty$

$\underline{\Pi}^{\mathrm{MAX-2MIN}}$

$$\max_{\lambda, q \geq 0, t} \sum_v \mu(v)\lambda(v) + \sum_x \rho(x)\Xi(x)$$

$$\text{s.t. } \Sigma q(x) \leq 1 \ \forall x; [\gamma(x)]$$

$$\lambda(v) + \Xi(x) \leq v \cdot \nabla q(x) \ \forall x, v; [\sigma(x, v)]$$

$\overline{\Pi}^{\mathrm{MIN-2MAX}}$

$$\min_{\sigma \geq 0, \gamma \geq 0, w} \sum_x \gamma(x)$$

$$\text{s.t. } \sum_x \sigma(x, v) = \mu(v) \ \forall v; [\Xi(x)]$$

$$\sum_v \sigma(x, v) = \rho(x) \ \forall x; [\lambda(v)]$$

$$w_i(x) = \frac{1}{\rho(x)} \sum_v v_i \sigma(x, v) \ \forall i, x;$$

$$\gamma(x) \geq \rho(x)[w_i(x) - \nabla_i w_i(x)] \ \forall i, x; [q_i(x)]$$

- Suppose the two programs are an exact dual pair as $k \to \infty$
- Then at the optimal, complementary-slackness conditions should hold

Binary common values

- Suppose $v_i \in \{0, 1\}$, and $\mu(\{v_1 = v_2 = \cdots = v_N\}) = 1$.
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- That is, the seller only knows the expected common value.
- As $k \to \infty$, $\rho_i(x_i) = \exp(-x_i)$.
- Value function:

$$w_i^*(x) = \begin{cases} C \exp(\Sigma x) & \Sigma x < x^* \\ 1 & \Sigma x \geq x^* \end{cases}$$

- All bidders have the same virtual value, which is 0 if $\Sigma x < x^*$ and is 1 otherwise
- The last constraint in $\underline{\Pi}^{\mathrm{MAX-2MIN}}$ is always binding, so $q_i^*(x)$ is free to be interior

MAX-2MIN mechanisms

- w^* implies:
  - $\sigma^*(x, v) > 0$ for both $v = \vec{0}$ and $v = \vec{1}$ when $\Sigma x < x^*$
  - $\sigma^*(x, v) > 0$ only for $v = \vec{1}$ when $\Sigma x \geq x^*$
- By complementary-slackness:
  - $\lambda^*(v) + \Xi^*(x) \leq v \cdot \nabla q^*(x)$ binds for both v when $\Sigma x < x^*$
  - it only binds for $v = \vec{1}$ when $\Sigma x \geq x^*$

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  - it only binds for $v = \vec{1}$ when $\Sigma x \geq x^*$
- With some additional regularity and boundary conditions, an allocation q is MAX-2MIN optimal as long as $\nabla_i q(x) = 1/x^*$ $(< 1/x^*)$ if $\Sigma x < x^*$ $(\geq x^*)$
- For example, these conditions are satisfied by the **proportional allocation**:

$$q_i^*(x) = \frac{x_i}{\max\{x^*, \Sigma x\}}$$

Transfers

- We show that for any such allocation,

$$\Xi^*(x) = \nabla \cdot q^*(x) - \lambda^*(\vec{1})$$

satisfies $\int_{\mathbb{R}_+^N} \Xi^*(x)\rho(dx) = 0$

- As a result, there always exists a transfer rules that solves

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- As a result, there always exists a transfer rules that solves

$$\Xi^*(x) = \nabla \cdot t(x) - \Sigma t(x)$$

- For the proportional allocation q^*, there is a transfer that solves this equation and has the proportional form:

$$t_i^*(x) = q_i^*(x)T^*(\Sigma x),$$

where T^* solves an ODE

- We call this MAX-2MIN mechanism **proportional auction**.

Truthful equilibrium

- In the finite bounding programs, we use a revelation principle, so the strategies are truthful/obedient
- Another heuristic for the continuum limit is that truthtelling/obedience should be locally optimal at the saddle point
- In fact, we show directly that the truthful/obedient strategies are an **equilibrium** for the saddle point we construct

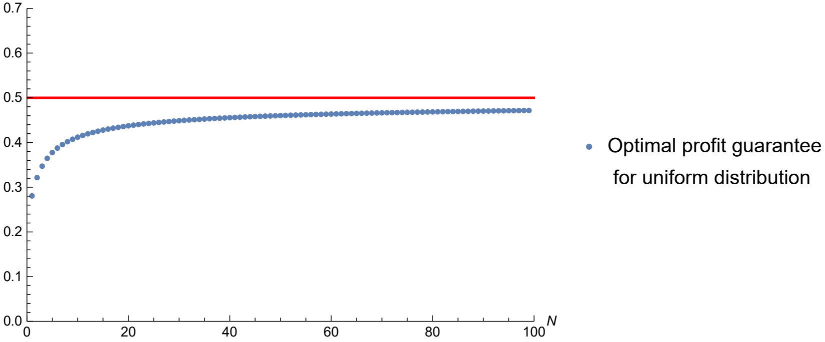
Robustness of proportional auction

- If the value is common but not binary, proportional auction remains MAX-2MIN optimal.
- If each bidder's expected value is known and the same, but the correlation and information structure are unknown, proportional auction remains MAX-2MIN optimal.
  - Common value is the worst-case (MIN-2MAX) information structure.

Common value full surplus extraction

Proposition

The profit guarantee of the proportional auction converges to the full surplus (expected common value) as $N \to \infty$ *at the rate of* $\frac{1}{\sqrt{N}}$.



Literature

- Bergemann, Brooks and Morris (2017), Brooks and Du (2021a,b,c)
- Robustness to correlation under private value: He and Li (2020), Zhang (2021)
- Distributional robustness under private value: Che (2020)
- Robustness to strategic uncertainty: Yamashita (2015)
- Robustness to resale opportunity: Carroll and Segal (2018)

Conclusion

- Linear programs that compute the optimal profit guarantee over all information structures and equilibria
- In many cases, the optimal (MAX-2MIN) mechanism is the proportional auction
- Guarantees full surplus extraction in common value setting with large markets
- Open questions:
  - Lower bounds on information? Private values?
  - Simple mechanisms that guarantee a good approximation of the full surplus?
  - Other welfare criteria? (Minmax regret for social surplus?)
  - Relax equilibrium assumption? (Rationalizability? Adaptive agents?)