

Online Appendix for “Robust Predictions with Bounded Information”^{*}

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1 Maxmin mechanism design

1.1 A general result

As a further application of our results on $F_{f,\epsilon}$ -refined BCE in linear games, we will consider a variation on the informationally robust mechanism design model of Brooks and Du (2024b), but where we now posit that the players have limited information about the state. Following Section 5.2, we consider the special case in which the base information structure is uninformative. Suppose a mechanism designer wants to maximize the mechanism’s performance guarantee over the information structures subject to an upper bound on the agents’ information. The designer controls an outcome $\omega \in \Omega$. Suppose the agents’ and designer’s utilities, $\tilde{u}_i(\omega, \theta)$ and $\tilde{w}(\omega, \theta)$, are linear in the state $\theta \in \Theta$ for each fixed ω , where Θ is a compact and convex subset of a normed vector space.

A mechanism is a tuple $M = (A, m)$ where $A = \prod_{i=1,\dots,N} A_i$ and $m : A \rightarrow \Delta(\Omega)$. A mechanism M induces a game where $u_i(a, \theta; M) = \sum_{\omega \in \Omega} m(\omega \mid a) \tilde{u}_i(\omega, \theta)$ and $w(a, \theta; M) = \sum_{\omega \in \Omega} m(\omega \mid a) \tilde{w}(\omega, \theta)$. A mechanism M is *participation secure* if for every player i there exists an action $0 \in A_i$ such that $u_i(0, a_{-i}, \theta; M) \geq 0$ for all $a_{-i} \in A_{-i}$ and $\theta \in \Theta$. Recall our definitions of $W(\phi; M)$ and $\text{BCE}(\mu; M)$ where we identify the game with its underlying mechanism M .

Following Brooks and Du (2024b), we consider action spaces parameterized by an integer k , where the k th action space has $k^2 + 1$ actions that are labeled $A_i = \{0, 1/k, 2/k, \dots, k\}$.

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Let $\mathcal{M}_k^0$ be the set of participation-secure mechanisms on A where 0 is a participation secure action for every player.

For a given $\mu \in \Delta(\Theta)$ and $\epsilon \geq 0$, the performance guarantee of a mechanism M over $F_{f,\epsilon}$ -refined BCE is

$$\min_{\phi \in F_{f,\epsilon}(A) \cap \text{BCE}(\mu; M)} W(\phi; M). \quad (1)$$

Thus, the guarantee-maximizing mechanism in $\mathcal{M}_k^0$ solves

$$\max_{M=(A,m) \in \mathcal{M}_k^0} \min_{\phi \in F_{f,\epsilon}(A) \cap \text{BCE}(\mu; M)} W(\phi; M). \quad (2)$$

In general, the set P_μ in Theorem 6 will not be convex, since for each prior in P_μ the support can be arbitrary but with at most $|A|$ elements. Nonetheless, it is without loss to convexify P_μ , as shown in the following proposition (whose proof is in the Section 1.3). Given a set X in a linear space, we denote its convex hull by $\text{conv } X$. Let $C = \text{conv}(\text{supp } \mu)$, where $\text{supp } \mu \subseteq \Theta$ is the support of μ and is assumed to be finite, and let $\overline{\text{conv}} P_\mu$ be the closure of $\text{conv } P_\mu$ in $\Delta(C)$ endowed the topology of weak convergence. Since C is compact, so are $\Delta(C)$ and $\overline{\text{conv}} P_\mu$. Clearly $\overline{\text{conv}} P_\mu$ is convex as well.

Proposition 1. *The value of problem (10) in Theorem 6 is equal to*

$$\min_{\mu' \in \overline{\text{conv}} P_\mu} \min_{\phi \in \text{BCE}(\mu')} W(\phi). \quad (3)$$

By Theorem 6 and Proposition 1, problem (2) is equivalent to:

$$\max_{M \in \mathcal{M}_k^0} \min_{\mu' \in \overline{\text{conv}} P_\mu} \min_{\phi \in \text{BCE}(\mu'; M)} W(\phi; M). \quad (4)$$

It is natural to ask how this program is related to

$$\min_{\mu' \in \overline{\text{conv}} P_\mu} \max_{M \in \mathcal{M}_k^0} \min_{\phi \in \text{BCE}(\mu'; M)} W(\phi; M). \quad (5)$$

If these two programs have the same value, then we would know that the solution to (4) reduces to the solution of the analogous maxmin problem with a different prior and no information constraints. This would be especially useful in cases where the solution to the maxmin problem with a fixed prior is known for all priors, such as in the common-value auction model studied in Brooks and Du (2021).

It is immediate that (4) is less than or equal to (5). However, it does not follow from the standard minimax theorem that (4) is equal to (5): for a fixed μ' , $\min_{\phi \in \text{BCE}(\mu'; M)} W(\phi; M)$ is generally neither a concave nor convex function of M .

On the other hand one can bound the gap between (4) and (5) by applying the bounding programs from Brooks and Du (2024b). In particular, Theorem 1 of Brooks and Du (2024b) shows that (4) is at least

$$\max_{M=(A,m) \in \mathcal{M}_k^0} \min_{\mu' \in \overline{\text{conv}} P_\mu} \sum_{\theta \in \Theta} \mu'(\theta) \min_a \sum_{\omega \in \Omega} \left[\tilde{w}(\omega, \theta) m(\omega|a) + \sum_{i=1, \dots, N} \tilde{u}_i(\omega, \theta) \nabla_i^+ m(\omega|a) \right], \quad (6)$$

where

$$\nabla_i^+ f(a) = \begin{cases} (k-1)(f(a_i + 1/k, a_{-i}) - f(a)) & \text{if } a_i < k; \\ 0 & \text{if } a_i = k. \end{cases}$$

The inner function in (6) is clearly linear in μ' for a fixed m and concave in m for a fixed μ' . Thus, by Sion's minimax theorem, (6) is equal to

$$\min_{\mu' \in \overline{\text{conv}} P_\mu} \max_{(A,m) \in \mathcal{M}_k^0} \sum_{\theta \in \Theta} \mu'(\theta) \min_a \sum_{\omega \in \Omega} \left[\tilde{w}(\omega, \theta) m(\omega|a) + \sum_{i=1, \dots, N} \tilde{u}_i(\omega, \theta) \nabla_i^+ m(\omega|a) \right]. \quad (7)$$

Moreover, (5) is clearly less than or equal to

$$\min_{\mu' \in \overline{\text{conv}} P_\mu} \min_{I=(S,\sigma): \sigma_\Theta = \mu'} \max_{M \in \mathcal{M}_k^0} \max_{\phi \in E_I(M)} W(\phi; M),$$

where we again identify M with the induced game structure. Brooks and Du (2024b) refer to the inner double maximand as the *potential* of an information structure I . By Theorem 1 of Brooks and Du (2024b), the minimum potential across all I with prior μ is at most

$$\min_{\mu' \in \overline{\text{conv}} P_\mu} \min_{\phi \in \Delta(A \times \Theta): \phi_\Theta = \mu'} \sum_{a \in A} \max_{\omega \in \Omega} \sum_{\theta \in \Theta} \left[\tilde{w}(\omega, \theta) \phi(a, \theta) - \sum_{i=1, \dots, N} \tilde{u}_i(\omega, \theta) \tilde{\nabla}_i^+ \phi(a, \theta) \right], \quad (8)$$

where

$$\tilde{\nabla}_i^+ f(a) = \begin{cases} -f(k, a_{-i}) & \text{if } a_i = k; \\ f(k, a_{-i}) - kf(k-1/k, a_{-i}) & \text{if } a_i = k-1/k; \\ k(f(a_i + 1/k, a_{-i}) - f(a)) & \text{otherwise.} \end{cases}$$

Thus, we conclude that the value of (7) is less than that of (4), which is less than that of (5), which is in turn less than (8). This proves the following result:

Theorem 1. *Let (μ', m) be a Nash equilibrium of the zero-sum game in (7). Then the guarantee of mechanism M over $F_{f,\epsilon}$ -refined BCE, given by (1), is at least the value of (7). Moreover, the guarantee of any mechanism is at most the value of problem (8).*

Thus, if the values of (7) and (8) are equal, then mechanisms maximizing (7) must approximately maximize the guarantee. In fact, Brooks and Du (2024a) have shown that the difference between the inner max in (7) and the inner min in (8) must tend to 0 as $k \rightarrow \infty$ for each μ' .

1.2 Application: Common value auction design

We simulated solutions of the programs (7) and (8) for the common value auction model studied in Brooks and Du (2021). In particular, Θ is a finite subset of $\mathbb{R}$. The outcome $\omega = (q, t) \in \mathbb{R}_+^N \times \mathbb{R}^N$ is a collection of allocations and transfers, where $\sum_{i=1,\dots,N} q_i = 1$.¹ Player i 's utility is

$$\tilde{u}_i(q, t, \theta) = \theta q_i - t_i.$$

The designer's objective is revenue maximization:

$$\tilde{w}(q, t, \theta) = \sum_{i=1,\dots,N} t_i.$$

Thus, the program (7) is a lower bound on the maximum guarantee for revenue across all mechanisms that always sell the good, and the program (8) is an upper bound on the minimum potential for revenue across all information structures, again assuming that the good must be sold.

We first solved programs (7) and (8) under the total variation information constraints when $k = 15$, meaning that each player has 226 actions, and μ is the uniform distribution on $\Theta = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ (as in Section 4).

The results of the simulations are depicted in Figures 2–4. Figure 2 depicts the information structures that minimize the upper bound on the potential (8). On the left-hand side is the joint distribution of the players' signals, and on the right-hand side is the players' expectation of the value conditional on the signal profile. There are two notable features: First, the likelihood of a signal profile only depends on the sum of the signals. While this is

¹This corresponds to the “must-sell” case studied in Brooks and Du (2021).

again not self-evident from the figure, it is also easily verified that the distribution of signals is independent, and these two properties together imply that the signals are iid exponential random variables. Second, the interim expected value is a non-decreasing function of the sum of the players' signals.

This is the same structure as identified in Brooks and Du (2021) for the potential minimizing information structure with a fixed prior, which coincides with the information in Figure 2 when $\epsilon = 2$.² In particular, the signals are iid exponential and the interim value is a non-decreasing function of the sum of the signals. The only difference is in the particular interim value function. Without any constraints on the players' information, the sum of the signals fully "reveals" the value, meaning that the distribution of the interim value is equal to the distribution of the ex post value. However, with an upper bound on information, the sum of the signals is only a noisy signal about the value. The particular noisy interim value distributions, which are mean-preserving contractions of the true prior, are depicted in the left panel of Figure 3. Consistent with Theorems 6 and 1, and according to the limit analysis of Brooks and Du (2021), this "noisy" interim value function would be the potential minimizing value function, without any constraints on the players' information, if we treated the interim value as the ex post value, and replaced the true prior with the corresponding contracted prior in Figure 3.

We obtain analogous results for the mechanisms that maximize the lower bound on the guarantee, depicted in Figure 4. In fact, for each value of ϵ , the computed allocations look nearly identical, and are close to the proportional allocation $q_i(a) = a_i/\Sigma a$, where $\Sigma a = a_1 + \dots + a_N$. Consistent with the results of Brooks and Du (2021), there seem to be many optimal transfer rules. In our simulations, we selected a particular transfer rule by imposing an additional constraint, that the aggregate transfer $\sum_{i=1,\dots,N} t_i(a)$ only depends on the sum of the actions Σa . Adding this constraint did not change the optimal value, and it resulted in transfer rules of the form $t_i(a) = T(\Sigma a)a_i/\Sigma a$, where T depends on the contracted prior and is plotted in the right panel of Figure 3. When $\epsilon = 2$, the aggregate transfer coincides with that in Brooks and Du (2021). When $\epsilon = 0$, the aggregate transfer is close to 0.5 for any positive aggregate action; the aggregate transfer is not exactly 0.5 since we use a discrete approximation of $k = 15$; we can see in Figure 1 that due to the discrete approximation, the revenue guarantee is close to but strictly less than 0.5 when $\epsilon = 0$, even though in this case the designer knows that all agents have no information about

²As with our discussion of the first-price auction, the analysis of Brooks and Du (2021) is primarily in a limit continuum model, although Section 5 of that paper shows that the potential minimizing information structures can be approached with finite information structures that have the same qualitative features as in the simulations depicted in Figure 2. A similar comment applies to the mechanisms depicted in Figure 4.

the common value. In sum, the mechanisms from the simulations have the form of the proportional auctions described in Brooks and Du (2021) and calibrated to the contracted priors of Figure 3.

The optimal values of programs (7) and (8) are plotted in Figure 1, along with that of the first-price auctions from Section 4. We see that the upper and lower bounds are close, and the lower bound is significantly larger than the revenue guarantee from the first-price auction when $\epsilon > 0$. Thus, the proportional auctions in Figure 4 significantly outperform first-price auction in terms of the revenue guarantee.

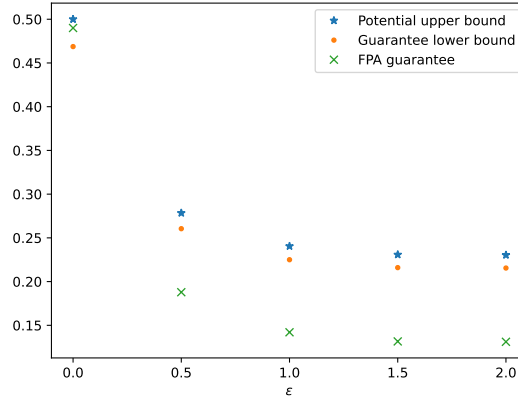


Figure 1: Revenue guarantees and potentials from maxmin common value auctions and first-price auctions under total variation information constraints.

Finally, we depict the corresponding pictures under the Kullback-Leibler information constraints in Figure 5, with the same piecewise linear approximation as in Section 4. We see that the interim value distributions, aggregate transfers, and revenues are qualitatively similar to those under total variation information constraints, though just as with the first-price auction, the interim value distributions tend to be smoother with Kullback-Leibler information constraints than with total variation distance.

The take-away from this application is that it is possible to combine the insights of this paper with the informationally robust mechanism design approach of Brooks and Du (2024b), in order to obtain new insights about informationally robust mechanisms when the players have bounded information about the state. This approach is especially powerful when the agents' preferences over the mechanism's outcome are linear in the state.

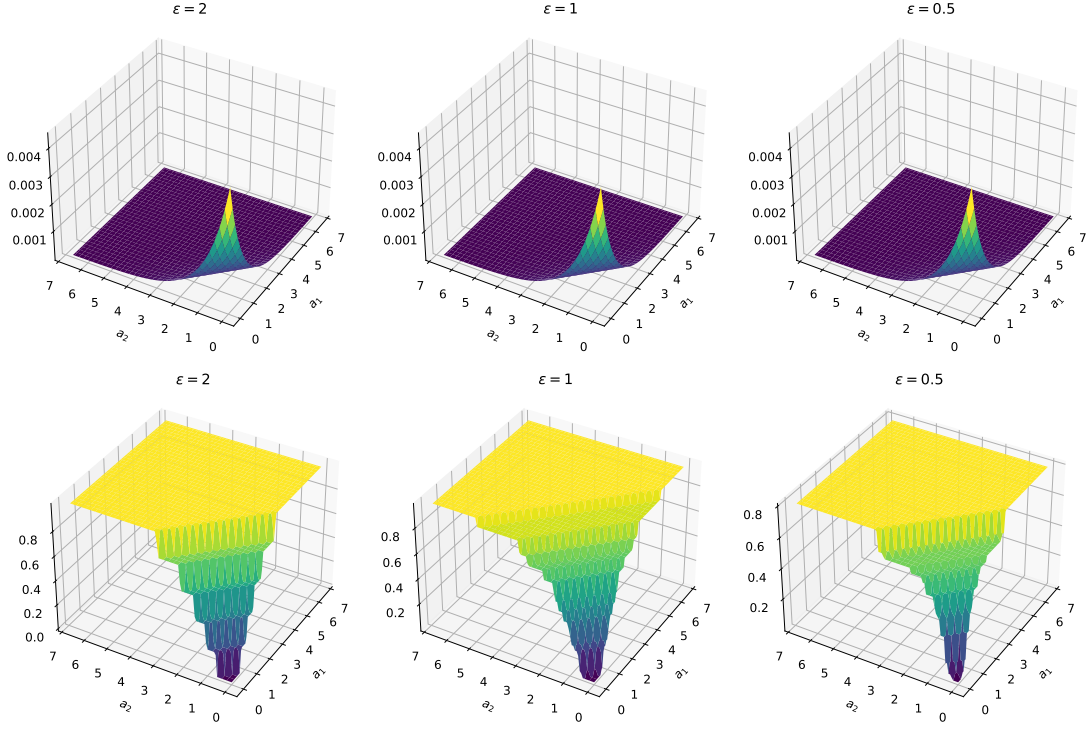


Figure 2: Information structures from program (8) under total variation information constraints; top row is the marginal distribution of the signal profiles, and bottom row is the interim value as a function of the signal profile.

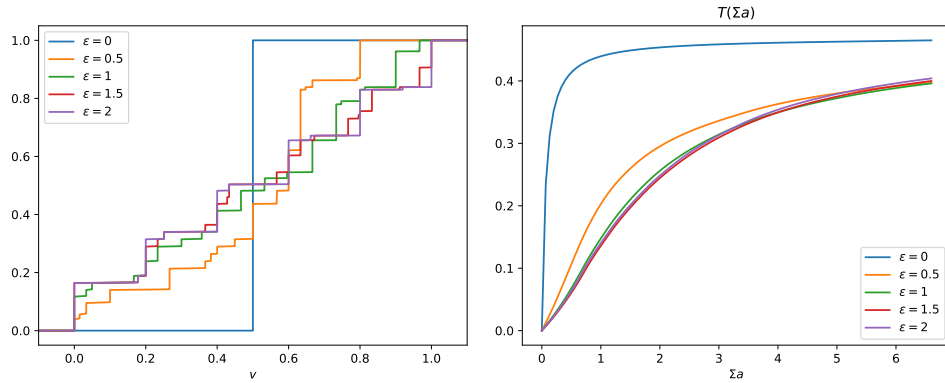


Figure 3: The interim value distributions and aggregate transfers from maxmin common value auctions under total variation information constraints.

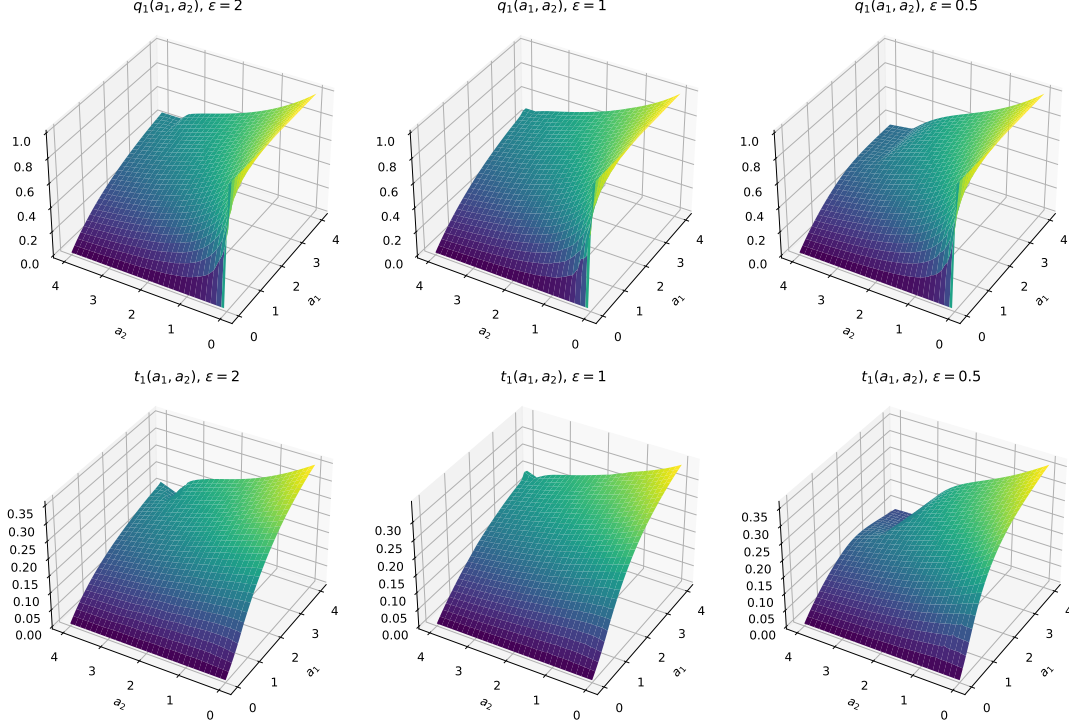


Figure 4: Maxmin common value auctions under total variation information constraints.

1.3 Proof of Proposition 1

To prove Proposition 1, we first note that when we pool together the conditional distributions of the state across the action profiles, the correlation between the state and action profile is reduced, and hence the f -information is also reduced:

Lemma 1. *For a $\phi \in \Delta(A \times \Theta)$, suppose $\beta = \phi_A$ places positive probability on a^1 and $a^2 \in A$. Let $\phi' \in \Delta(A \times \Theta)$ be such that $\phi'_A = \beta$ and*

$$\phi'(\theta | a) = \begin{cases} \phi(\theta | a) & a \notin \{a^1, a^2\}; \\ \phi(\theta | a^1) \frac{\beta(a^1)}{\beta(a^1) + \beta(a^2)} + \phi(\theta | a^2) \frac{\beta(a^2)}{\beta(a^1) + \beta(a^2)} & a \in \{a^1, a^2\}. \end{cases}$$

Then we have $D_f(\phi' \| \beta \otimes \mu) \leq D_f(\phi \| \beta \otimes \mu)$.

Proof. This follows from the data processing inequality (Polyanski and Wu, 2023, Theorem 7.4) using the kernel:

$$b(a', \theta' | a, \theta) = \begin{cases} 1 & a = a' \notin \{a^1, a^2\} \text{ and } \theta = \theta' \\ \frac{\beta(a')}{\beta(a^1) + \beta(a^2)} & a', a \in \{a^1, a^2\} \text{ and } \theta = \theta' \\ 0 & \text{else} \end{cases}$$

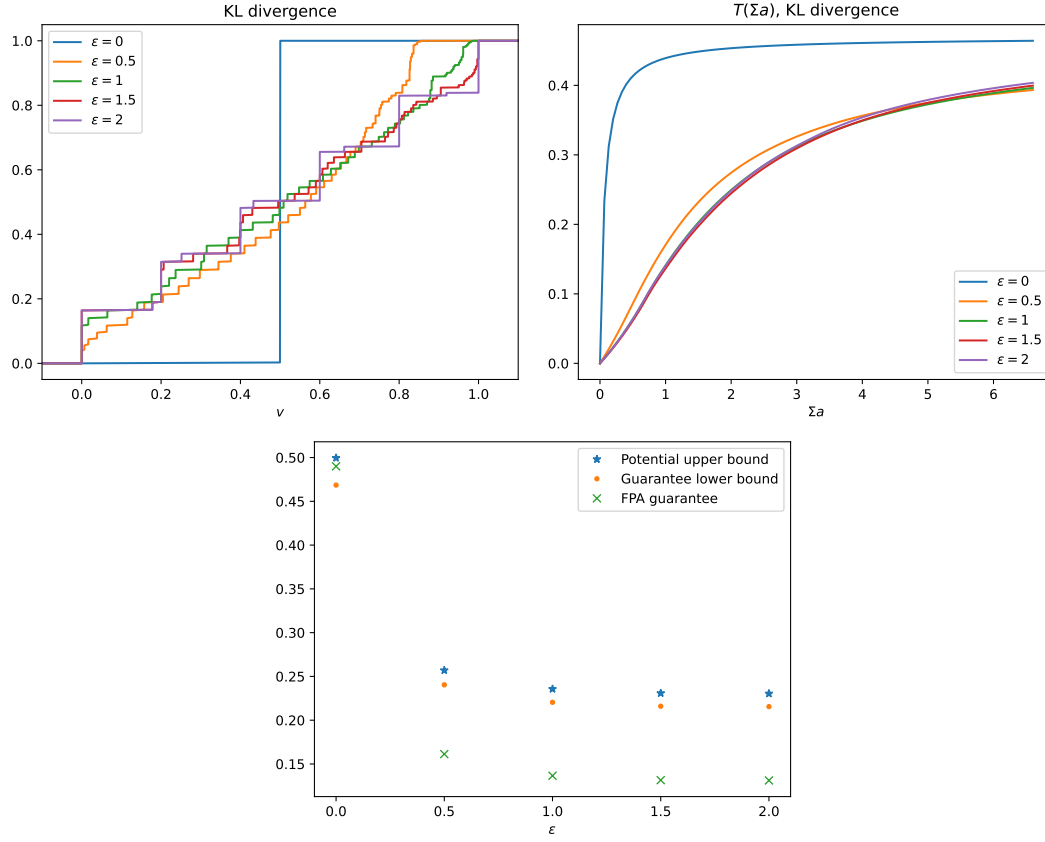


Figure 5: The interim value distributions, aggregate transfers, and revenues from maxmin common value auctions under Kullback-Leibler information constraints.

□

Proof of Proposition 1. Suppose $\mu' = \alpha\tilde{\mu} + (1 - \alpha)\hat{\mu}$, where $\tilde{\mu}, \hat{\mu} \in P_\mu$ and $\alpha \in [0, 1]$ (for notational simplicity, we assume a convex combination of only two elements in P_μ). By the definition of P_μ , there exist $\tilde{\phi}, \hat{\phi} \in \Delta(A \times \Theta)$ such that $\nu(\tilde{\phi}) = \tilde{\mu}$, $\nu(\hat{\phi}) = \hat{\mu}$, $D_f(\tilde{\phi} \parallel \tilde{\beta} \otimes \mu) \leq \epsilon$ and $D_f(\hat{\phi} \parallel \hat{\beta} \otimes \mu) \leq \epsilon$, where $\tilde{\beta}$ and $\hat{\beta}$ are the marginal distributions of $\tilde{\phi}$ and $\hat{\phi}$ over A , respectively. By Lemma 1, we can assume without loss that $\tilde{\phi}(a, \theta) = \tilde{\beta}(a)\tilde{\rho}(\theta \mid \tilde{\eta}(a))$ and $\hat{\phi}(a, \theta) = \hat{\beta}(a)\hat{\rho}(\theta \mid \hat{\eta}(a))$, where $\tilde{\eta}(a)$ and $\hat{\eta}(a)$ are the interim state given a , and $\tilde{\rho}$ and $\hat{\rho}$ are the unbiased noises conditional on the interim state.

Note that

$$\begin{aligned} D_f(\tilde{\phi} \parallel \tilde{\beta} \otimes \mu) &= \sum_{a \in A, \theta \in \Theta} f\left(\frac{\tilde{\phi}(a, \theta)}{\tilde{\beta}(a)\mu(\theta)}\right) \tilde{\beta}(a)\mu(\theta) \\ &= \sum_{a \in A, \theta \in \Theta} f\left(\frac{\tilde{\rho}(\theta \mid \tilde{\eta}(a))}{\mu(\theta)}\right) \tilde{\beta}(a)\mu(\theta), \\ &= \sum_{\theta \in \Theta, \theta' \in \Theta} f\left(\frac{\tilde{\rho}(\theta \mid \theta')}{\mu(\theta)}\right) \tilde{\mu}(\theta')\mu(\theta), \end{aligned}$$

and analogously

$$D_f(\hat{\phi} \parallel \hat{\beta} \otimes \mu) = \sum_{\theta \in \Theta, \theta' \in \Theta} f\left(\frac{\hat{\rho}(\theta \mid \theta')}{\mu(\theta)}\right) \hat{\mu}(\theta')\mu(\theta),$$

Therefore, by convexity of f , we have

$$\begin{aligned} \epsilon &\geq \alpha D_f(\tilde{\phi} \parallel \tilde{\beta} \otimes \mu) + (1 - \alpha) D_f(\hat{\phi} \parallel \hat{\beta} \otimes \mu) \\ &= \sum_{\theta \in \Theta} \sum_{\theta' \in \Theta} f\left(\frac{\tilde{\rho}(\theta \mid \theta')}{\mu(\theta)}\right) \alpha \tilde{\mu}(\theta')\mu(\theta) + \sum_{\theta \in \Theta} \sum_{\theta' \in \Theta} f\left(\frac{\hat{\rho}(\theta \mid \theta')}{\mu(\theta)}\right) (1 - \alpha) \hat{\mu}(\theta')\mu(\theta) \\ &\geq \sum_{\theta \in \Theta} \sum_{\theta' \in \Theta} f\left(\frac{\rho'(\theta \mid \theta')}{\mu(\theta)}\right) \mu'(\theta')\mu(\theta), \end{aligned} \tag{9}$$

where

$$\rho'(\theta \mid \theta') = \tilde{\rho}(\theta \mid \theta') \frac{\alpha \tilde{\mu}(\theta')}{\mu'(\theta')} + \hat{\rho}(\theta \mid \theta') \frac{(1 - \alpha) \hat{\mu}(\theta')}{\mu'(\theta')}.$$

Now, suppose $\phi' \in \text{BCE}(\mu')$, and let $\mu'' = \nu(\phi')$. Effectively we need to show that $\mu'' \in P_\mu$. Without loss suppose there exists noise ρ'' such that $\phi'(a, \theta') = \beta'(a)\rho''(\theta' \mid \eta'(a))$ for all $a \in A$ and $\theta' \in \Theta$. Add noise ρ' to ϕ' to arrive at an outcome $\phi \in \text{BCE}(\mu)$: $\phi(a, \theta) = \sum_{\theta' \in \Theta} \phi'(a, \theta')\rho'(\theta \mid \theta')$. Note that the marginal distributions of ϕ and ϕ' over A

are the same: $\beta = \beta'$. We compute

$$\begin{aligned}
D_f(\phi \parallel \beta \otimes \mu) &= \sum_{\theta \in \Theta} \sum_{\theta'' \in \Theta} f\left(\frac{\sum_{\theta' \in \Theta} \rho''(\theta' \mid \theta'') \rho'(\theta \mid \theta')}{\mu(\theta)}\right) \mu''(\theta'') \mu(\theta) \\
&\leq \sum_{\theta \in \Theta} \sum_{\theta'' \in \Theta} \sum_{\theta' \in \Theta} f\left(\frac{\rho'(\theta \mid \theta')}{\mu(\theta)}\right) \rho''(\theta' \mid \theta'') \mu''(\theta'') \mu(\theta) \\
&= \sum_{\theta \in \Theta} \sum_{\theta' \in \Theta} f\left(\frac{\rho'(\theta \mid \theta')}{\mu(\theta)}\right) \mu'(\theta') \mu(\theta) \\
&\leq \epsilon,
\end{aligned}$$

where the last inequality follows from (9). Thus, we have $\mu'' \in P_\mu$, and $W(\phi') = W(\phi)$ is greater than or equal to the optimal value of problem (10).

Thus, for an arbitrary element μ' in the convex hull of P_μ , the inner minimization over $\phi \in \text{BCE}(\mu')$ in (3) is greater than or equal to the optimal value of problem (10). Since the inner minimization over $\phi \in \text{BCE}(\mu')$ in (3) is a continuous function of μ' by Berge's Maximum Theorem, we conclude that the optimal value of problem (3) is greater than or equal to that of problem (10).

Moreover, since $P_\mu \subseteq \overline{\text{conv}} P_\mu$, the optimal value of problem (3) is obviously less than or equal to that of problem (10). This proves the proposition. $\square$

2 Relations on information structures

In this Appendix, we discuss the connection between our results and the literature on comparisons of information structures. Suppose the base information structure $\underline{I} = (\underline{S}, \underline{\sigma})$ is uninformative. The notions of individual garbling and coordinated individual garbling have been previously introduced in the literature, notably in Gossner (2000) and Lehrer, Rosenberg, and Shmaya (2013). We discuss each of these in turn.

Gossner (2000) asks the question: Given information structures I and I' , when is it the case that $E_I(G) \subseteq E_{I'}(G)$ for all G ? His main result shows that this is the case if and only if, in our terminology, I is a coordinated individual garbling of I' .³ Thus, coordinated individual garblings represent a natural preorder on information structures.

Gossner's setup is different from ours in that he allows for compact and continuous games. As a result, his result does not imply ours, nor do our results imply his. We now state an analogue of Gossner's theorem for our finite setting:

³Gossner would say that I is "faithfully reproduced" from I' .

Proposition 2. *I is a coordinated individual garbling of I' if and only if $E_I(G) \subseteq E_{I'}(G)$ for all G .*

Proof of Proposition 2. For the if direction, let G be the separation game and ϕ the equilibrium outcome, as in Lemma 1. By hypothesis, $\phi \in E_{I'}(G)$, so by Lemma 1, I is a coordinated individual garbling of I' .

We now prove the only if direction. Suppose $I = (S, \sigma)$ is a coordinated individual garbling of $I' = (S', \sigma')$, where $b' \in B(S, S')$ is the individual garbling that satisfies (2). Let $G = (A, u)$ and $\phi \in E_I(G)$, and b the strategies that induce ϕ . We claim that the strategies

$$\hat{b}_i(a_i|s'_i) = \sum_{s_i \in S_i} b_i(a_i|s_i) b'_i(s_i|s'_i)$$

are an equilibrium of (I', G) that induces ϕ . To see that $\hat{b}$ is an equilibrium, first note that

$$\begin{aligned} \hat{b}_{-i}(a_{-i}|s'_{-i}) &\equiv \prod_{j \neq i} \sum_{s_j \in S_j} b_j(a_j|s_j) b'_j(s_j|s'_j) \\ &= \sum_{s_{-i} \in S_{-i}} b_{-i}(a_{-i}|s_{-i}) b'_{-i}(s_{-i}|s'_{-i}), \end{aligned}$$

where analogously $b_{-i}(a_{-i}|s_{-i}) \equiv \prod_{j \neq i} b_j(a_j|s_j)$, and so on. Now, by equation (2), for every s_i and $s'_i \in S'_i$ such that $b'_i(s_i|s'_i) > 0$, there exists an $\alpha > 0$ such that for all (s_{-i}, θ) ,

$$\alpha \sigma(s_i, s_{-i}, \theta) = b'_i(s_i|s'_i) \sum_{s'_{-i}} \left(\prod_{j \neq i} b'_j(s_j|s'_j) \right) \sigma'(s'_i, s'_{-i}, \theta).$$

Hence, for any i , s'_i , and a'_i ,

$$\begin{aligned} &\sum_{a_{-i}} \sum_{s'_{-i} \in S'_{-i}, \theta \in \Theta} \hat{b}_{-i}(a_{-i}|s'_{-i}) \sigma'(s'_i, s'_{-i}, \theta) u_i(a'_i, a_{-i}, \theta) \\ &= \sum_{a_{-i}} \sum_{s'_{-i} \in S'_{-i}, \theta \in \Theta} \sum_{s_{-i} \in S_{-i}} b_{-i}(a_{-i}|s_{-i}) b'_{-i}(s_{-i}|s'_{-i}) \sigma'(s'_i, s'_{-i}, \theta) u_i(a'_i, a_{-i}, \theta) \\ &= \alpha \sum_{a_{-i}} \sum_{s_{-i} \in S_{-i}, \theta \in \Theta} b_{-i}(a_{-i}|s_{-i}) \sigma(s_i, s_{-i}, \theta) u_i(a'_i, a_{-i}, \theta), \end{aligned}$$

which is proportional to the interim expected utility of type s_i playing a'_i when others are using b_{-i} , in the information structure I . Thus, if a_i is a best response for some s_i with $b'_i(s_i|s'_i) > 0$, then it is a best response for s'_i as well. Since $\hat{b}_i(\cdot|s'_i)$ is supported on such a_i 's, and it follows that $\hat{b}_i$ is a best response to $\hat{b}_{-i}$. Finally, because b' is an individual garbling,

we have that

$$\begin{aligned}\sum_{s' \in S'} \hat{b}(a|s') \sigma'(s', \theta) &= \sum_{s \in S} b(a|s) \sum_{s' \in S'} b'(s|s') \sigma'(s', \theta) \\ &= \sum_{s \in S} b(a|s) \sigma(s, \theta) = \phi(a, \theta),\end{aligned}$$

so that $\hat{b}$ and I' induce ϕ . We conclude that $\phi \in E_{I'}(G)$, as desired. $\square$

The proof of the only if is morally the same as Gossner's: From the fact that I is a coordinated individual garbling of I , we know that the players can “simulate” I from I' in such a way that each player's garbled signal is sufficient for other players' garbled signals and the state. Thus, any equilibrium of I has an equivalent equilibrium of I' , where the players first simulate I and then play the given strategies on I .

The if direction of the proof of Proposition 2, however, is substantively different. Gossner also constructs a game that plays an analogous role as our separation game. However, in his game, players report signals as well as beliefs about others' signals (and the state, in his extension to incomplete information). The payoff for the belief is given by a log scoring rule, with the payoff defined to be $-\infty$ if a player assigns zero probability to the signals that are reported by the others. Because Gossner's game is not compact, an extra step is needed to approximate the obedient outcome of his separation game via compact games. He establishes that an approximate analogue of the coordinated individual garbling property holds, and he finally takes limits to conclude that it holds exactly. In comparison, the construction and analysis of our separation game is elementary. The game is finite, and no scoring rules, infinite payoffs, or approximations are needed.

In a slightly different but related direction, Lehrer, Rosenberg, and Shmaya (2013) study equivalence relations on information structures that arise from having the same set of equilibrium outcomes for all games, according to various equilibrium concepts, and including Bayes Nash equilibrium. Their main result for Bayes Nash equilibrium is that two information structures have the same equilibrium outcomes for all games if and only if they are individual garblings of one another. While it is not a primary objective of our paper, our investigation has led us to the observation that the equivalence relations of Lehrer, Rosenberg, and Shmaya (2013) are also equivalent to several other notions, which we now state.

We say that $I' = (S', \sigma')$ is a *reduction* of $I = (S, \sigma)$ if there are mappings $f_i : S_i \rightarrow S'_i$ for each i such that (i) if $f_i(s_i) = f_i(\hat{s}_i)$, then there exists an $\alpha \in \mathbb{R}$ such that for all

$(s_{-i}, \theta) \in S_{-i} \times \Theta$, we have

$$\sigma(s_i, s_{-i}, \theta) = \alpha \sigma(\hat{s}_i, s_{-i}, \theta),$$

and (ii) for any $(s', \theta) \in S' \times \Theta$, we have that

$$\sigma'(s', \theta) = \sum_{s \in f^{-1}(s')} \sigma(s, \theta).$$

In other words, I' is obtained from I by merging types that have the same interim beliefs. We say that I is *irreducible* if for every i , no two types have the same interim beliefs. We say that I is *reduction equivalent* to I' if there is an information structure I'' that is a reduction of both I and I' .

Proposition 3. *Given information structures I and I' , the following statements are equivalent:*

- (a) I and I' are individual garblings of each other.
- (b) I and I' are coordinated individual garblings of each other.
- (c) $E_I(G) = E_{I'}(G)$ for all G (equilibrium outcome equivalence).
- (d) $F_I(A) = F_{I'}(A)$ for all A (outcome equivalence).
- (e) I and I' are reduction equivalent.

Proof of Proposition 3. For the sake of completeness, we give a self-contained proof of the proposition, including those parts known from prior work.

(b) $\implies$ (a): Clearly, if I and I' are coordinated individual garblings of each other, then they are also individual garblings of each other.

(a) $\implies$ (d): Suppose $I = (S, \sigma)$ is an individual garbling of $I' = (S', \sigma')$, with garbling $b' \in B(S', S)$, and let $\phi \in F_I(A)$, induced by strategies $b' \in B(S, A)$. Define the strategies $\hat{b} \in B(S', A)$ by

$$\hat{b}_i(a_i | s'_i) = \sum_{s_i \in S_i} b_i(a_i | s_i) b'_i(s_i | s'_i).$$

Then the outcome ϕ' induced by $\hat{b}$ and I' is

$$\phi'(a, \theta) = \sum_{s' \in S'} \hat{b}(a | s') \sigma'(s', \theta)$$

$$\begin{aligned}
&= \sum_{s' \in S'} \prod_{i=1, \dots, N} \sum_{s_i \in S_i} b_i(a_i | s_i) b'_i(s_i | s'_i) \sigma'(s', \theta) \\
&= \sum_{s' \in S'} \sum_{s \in S} b(a | s) b'(s | s') \sigma'(s', \theta) \\
&= \sum_{s \in S} b(a | s) \sum_{s' \in S'} b'(s | s') \sigma'(s', \theta) \\
&= \sum_{s \in S} b(a | s) \sigma(s, \theta) \\
&= \phi(a, \theta)
\end{aligned}$$

so that $\phi \in F_{I'}(A)$. Since ϕ was arbitrary, we have $F_I(A) \subseteq F_{I'}(A)$. Reversing the roles of I and I' gives $F_{I'}(A) \subseteq F_I(A)$, so that I and I' are outcome equivalent, as desired.

(b) $\implies$ (c): This follows immediately from Proposition 2.

(c) $\implies$ (b): Let G be the separation game and ϕ the equilibrium outcome $\phi \in E_I(G)$ given in Lemma 1. By (c), $\phi \in E_{I'}(G)$, so by Lemma 1, I is a coordinated individual garbling of I' . Repeating this argument with I and I' reversed implies that I' is an individual garbling of I as well.

(c) $\implies$ (d): Fix an action space A and let $G = (A, u)$ where $u_i(a) = 0$ for all i and $a \in A$. From equilibrium outcome equivalence, we have that $E_I(G) = E_{I'}(G)$. But because players are indifferent between all actions, $E_I(G) = F_I(A)$ and $E_{I'}(G) = F_{I'}(A)$. We conclude that $F_I(A) = F_{I'}(A)$, i.e., I and I' are outcome equivalent.

(d) $\implies$ (e): Suppose that I and I' are outcome equivalent. Since I and I' are reduction equivalent to their respective reductions, it is without loss to assume that I and I' are irreducible. Now, clearly we have that $\sigma \in F_I(S)$ and $\sigma' \in F_{I'}(S')$. Outcome equivalence therefore implies that $\sigma \in F_{I'}(S)$ and $\sigma' \in F_I(S')$. Let b and b' be strategies such that (I, b) induce σ' and (I', b') induce σ . Define the Markov kernels $K_i : S_i \rightarrow \Delta(S'_i)$ according to

$$K_i(\hat{s}_i | s_i) = \sum_{s'_i \in S'_i} b'_i(\hat{s}_i | s'_i) b_i(s'_i | s_i).$$

Also define the product kernel $K(\hat{s} | s) = \prod_{i=1, \dots, N} K_i(\hat{s}_i | s_i)$. It follows from the fact that (I, b) induce σ' and (I', b') induce σ that σ is an invariant measure for K , in the sense that for all $(\hat{s}, \theta)$, we have

$$\sigma(\hat{s}, \theta) = \sum_{s \in S} K(\hat{s} | s) \sigma(s, \theta).$$

Now, let d_i such that the kernel $K_i^{d_i}$ is aperiodic, and let $d = \prod_{i=1,\dots,N} d_i$ (so that K_i^d and K^d are all aperiodic). Let P_i be the partition of S_i into communicating classes of K_i and let P be the partition of S into communicating classes of K . It is easy to see that $P = \prod_{i=1,\dots,N} P_i$. Note that because K_i^d is aperiodic, there is a unique invariant measure of K_i^d restricted to $p_i \in P_i$, which we denote by $\pi_i^{p_i}$. Similarly, if K^d is restricted to $p = \prod_{i=1,\dots,N} p_i \in P$, there is a unique invariant measure π^p , and since $\prod_{i=1,\dots,N} \pi_i^{p_i}$ is also an invariant of K^d on p , we must have $\pi^p = \prod_{i=1,\dots,N} \pi_i^{p_i}$. Hence, if we write

$$\sigma(p, \theta) = \sum_{s \in p} \sigma(s, \theta),$$

then for $s \in p \in P$, we have

$$\sigma(s, \theta) = \sigma(p, \theta) \pi^p(s).$$

Now, fix i , $s \in p \in P$, and $\theta \in \Theta$. Then

$$\begin{aligned} \sigma_i(s_{-i}, \theta | s_i) &= \frac{\sigma(s_i, s_{-i}, \theta)}{\sum_{s'_{-i} \in S_{-i}, \theta' \in \Theta} \sigma(s_i, s'_{-i}, \theta')} \\ &= \frac{\pi^p(s_i, s_{-i}) \sigma(p, \theta)}{\sum_{p'_{-i} \in P_{-i}, \theta' \in \Theta} \sigma(p_i, p'_{-i}, \theta') \sum_{s'_{-i} \in p'_{-i}} \pi^{p_i, p'_{-i}}(s_i, s'_{-i})} \\ &= \frac{\pi_i^{p_i}(s_i) \pi_{-i}^{p_{-i}}(s_{-i}) \sigma(p, \theta)}{\sum_{p'_{-i} \in P_{-i}, \theta' \in \Theta} \sigma(p_i, p'_{-i}, \theta') \sum_{s'_{-i} \in p'_{-i}} \pi_i^{p_i}(s_i) \pi_{-i}^{p'_{-i}}(s'_{-i})} \\ &= \frac{\pi_i^{p_i}(s_i) \pi_{-i}^{p_{-i}}(s_{-i}) \sigma(p, \theta)}{\pi_i^{p_i}(s_i) \sum_{p'_{-i} \in P_{-i}, \theta' \in \Theta} \sigma(p_i, p'_{-i}, \theta')} \\ &= \frac{\pi_{-i}^{p_{-i}}(s_{-i}) \sigma(p, \theta)}{\sum_{p'_{-i} \in P_{-i}, \theta' \in \Theta} \sigma(p_i, p'_{-i}, \theta')}. \end{aligned}$$

This expression depends on p_i but not on the particular $s_i \in p_i$. Hence, it must be that if $s_i, s'_i \in p_i$, then $\sigma_i(\cdot, \cdot | s_i) = \sigma_i(\cdot, \cdot | s'_i)$. From the hypothesis that I is irreducible, we conclude that $|p_i| = 1$, and thus $K^d(s|s) = 1$. This is possible only if b' is a pure strategy such that $b'(s|s') = 1$ for any s' such that $b(s'|s) > 0$. By a similar analysis, we conclude that b' is also pure. Hence, the function $f_i(s_i)$ defined according to $b_i(f_i(s_i)|s_i) = 1$ is a bijection from S_i to S'_i . This function satisfies (i) and (ii) in the definition of reduction, so that I' is a reduction of I , and vice versa.

(e) $\implies$ (c): We will show this for the special case in which $S' \subseteq S$, and $f_i(s''_i) = s''_i$ for all $s''_i \notin \{s_i, s'_i\}$ and $f_j(s_j) = s_j$ for all $s_j \in S_j$. In other words, exactly two signals are merged

for player i , and no signals are merged for players $j \neq i$. Without loss, we assume that s_i and s'_i arise with positive probability. Let $\sigma_i(s_i)$ denote the marginal probability of s_i , and let $\sigma_i(s_{-i}, \theta | s_i)$ denote the conditional distribution of (s_{-i}, θ) given s_i , and defined similarly for σ' . From the definition of reduction equivalence, we have that $\sigma'_j(\cdot, \cdot | f_j(s_j)) = \sigma_j(\cdot, \cdot | s_j)$ and $\sigma'_j(s'_j) = \sum_{s_j \in f_j^{-1}(s'_j)} \sigma_j(s_j)$ for all j and $s_j \in S_j$.

Now, let b and b' be strategies in (I, G) and (I', G) respectively, such that (i) $b'_j(s_j) = b_j(s_j)$ for all $j \neq i$ and s_j , (ii) $b'_i(s''_i) = b_i(s''_i)$ for all $s''_i \notin \{s_i, s'_i\}$, and (iii)

$$b'_i(\bar{s}_i) = \frac{1}{\sigma_i(s_i) + \sigma_i(s'_i)} (\sigma_i(s_i)b_i(s_i) + \sigma_i(s'_i)b_i(s'_i)). \quad (10)$$

where $\bar{s}_i = f_i(s_i) = f_i(s'_i)$. Then (I', b') induce the outcome

$$\begin{aligned} & \sum_{s' \in S} \sigma'(s', \theta) b'(a | s') \\ &= \sum_{s_{-i} \in S_{-i}} \left[\sum_{s''_i \notin \{s_i, s'_i\}} \sigma'(s''_i, s_{-i}, \theta) b'(a | s''_i, s_{-i}) + \sigma'_i(\hat{s}_i) \sigma'(s_{-i}, \theta | \hat{s}_i) b'(a | \hat{s}_i, s_{-i}) \right] \\ &= \sum_{s_{-i} \in S_{-i}} \left[\sum_{s''_i \notin \{s_i, s'_i\}} \sigma(s''_i, s_{-i}, \theta) b(a | s''_i, s_{-i}) + \sigma'_i(\hat{s}_i) b'_i(a_i | \hat{s}_i) \sigma'(s_{-i}, \theta | \hat{s}_i) \prod_{j \neq i} b'_{-j}(a_{-j} | s_{-j}) \right] \\ &= \sum_{s_{-i} \in S_{-i}} \left[\sum_{s''_i \notin \{s_i, s'_i\}} \sigma(s''_i, s_{-i}, \theta) b(a | s''_i, s_{-i}) + \left(\sum_{s''_i \in \{s_i, s'_i\}} \sigma_i(s''_i) b'_i(a_i | s''_i) \right) \sigma(s_{-i}, \theta | s''_i) \prod_{j \neq i} b_{-j}(a_{-j} | s_{-j}) \right] \\ &= \sum_{s_{-i} \in S_{-i}} \left[\sum_{s''_i \notin \{s_i, s'_i\}} \sigma(s''_i, s_{-i}, \theta) b(a | s''_i, s_{-i}) + \left(\sum_{s''_i \in \{s_i, s'_i\}} \sigma_i(s''_i) b_i(a_i | s''_i) \right) \sigma(s_{-i}, \theta | s''_i) \prod_{j \neq i} b_{-j}(a_{-j} | s_{-j}) \right] \\ &= \sum_{s \in S} \sigma(s, \theta) b(a | s) = \phi(a, \theta). \end{aligned}$$

Hence $U_i(b'; I', G) = U_i(b; I, G)$.

Now, we claim that if b and b' satisfy (i)–(iii) above, then b is an equilibrium if and only if b' is an equilibrium. We have already established that $U_j(b; I, G) = U_j(b'; I', G)$ for all j . If b' is not an equilibrium, then there exists j and $\hat{b}'_j$ such that $U_j(\hat{b}'_j, b'_{-j}; I', G) > U_j(b'; I', G)$. Now define $\hat{b}_j(s_j) = \hat{b}'_j(f_j(s_j))$. Then $(\hat{b}'_j, b'_{-j})$ and $(\hat{b}_j, b_{-j})$ satisfy (i)–(iii), so that $U_j(\hat{b}_j, b_{-j}; I, G) = U_j(\hat{b}'_j, b'_{-j}; I', G)$, so that $\hat{b}_j$ is a profitable deviation from (I, b) . Alternatively, if $\hat{b}_j$ is a profitable deviation from (I, b) , then we can define $\hat{b}'_j$ according to (10), so that (i)–(iii) are again satisfied for $(\hat{b}'_j, b'_{-j})$ and $(\hat{b}_j, b_{-j})$, and we similarly conclude

that $\hat{b}'_j$ is a profitable deviation from (I', b') . This completes the proof that I and I' are equilibrium outcome equivalent when I' is a reduction of I obtained by merging two signals.

Iterative application of this step (and relabeling of signals) then establishes that I is equilibrium outcome equivalent to I' if I' is a reduction of I . Now, if I and I' are reduction equivalent, then there exists an information structure I'' that is a reduction of I and I' . We therefore have $E_I(G) = E_{I''}(G) = E_{I'}(G)$.

We conclude that $(b) \iff (c) \implies (d) \implies (e) \implies (c)$. So that (b) – (e) are all equivalent. Finally, $(a) \implies (d)$ and $(b) \implies (a)$, so that (b) – (e) are all equivalent to (a) as well. $\square$

The proof of Proposition 3 shows that if I' is a reduction of I and I'' is a reduction of I' , then I'' is in fact a reduction of I . This implies that any for information structure I , there is a unique I' (up to a relabeling of signals) that is an irreducible reduction of I . Moreover, I' can be obtained “at one step”, by merging signals in I that have the same interim belief. In this sense, there is a simple finite procedure for determining if I and I' are equivalent, that eliminates the existential and universal quantifiers over individual garblings and games, respectively.

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