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RESEARCH ARTICLE

Representing type spaces as signal allocations

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Abstract
Consider a set of agents uncertain about the state in some finite state space Ω. A *type space* $(\boldsymbol{T}, Q)$ that describes the agents' information consists of a finite product set $\boldsymbol{T} = T_1 \times \cdots \times T_n$, and a probability distribution $Q \in \Delta(\Omega \times \boldsymbol{T})$. Alternatively, a *signal allocation* assigns to each agent i a *signal* π_i, a finite partition of $\Omega \times X$ where X is a measurable space endowed with a non-atomic probability measure. Every signal allocation induces a type space in which the types in T_i are the elements of π_i. We establish two results. First, every type space is equivalent to one that is induced by a signal allocation. Second, encoding of type spaces into signal allocations can be done myopically, one agent at a time.

Keywords Type spaces · Signals

JEL classification C70 · D82 · D83 · D85

1 Introduction

1.1 Informational environment

Fix a finite state space Ω and an interior prior $\mu_0 \in \Delta\Omega$. Suppose n agents have (potentially correlated) information about the state. There are two common approaches to formalizing such an informational environment.

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A *type space* $(\boldsymbol{T}, Q)$ consists of a finite product set $\boldsymbol{T} = T_1 \times \cdots \times T_n$, and a probability distribution $Q \in \Delta(\Omega \times \boldsymbol{T})$ for which the marginal of Q on Ω is μ_0. The interpretation is that agent i observes the realization $t_i \in T_i$. The correlation between t_i and $\omega \in \Omega$ allows the agent to make inferences about the state while the overall Q allows the agent to make inferences about the state and (first- and higher-order) beliefs of the other agents. We say two type spaces are equivalent, denoted $(T_1 \times \cdots \times T_n, Q) \sim \left(T_1' \times \cdots \times T_n', Q'\right)$, if there are bijections $\left\{f_i : T_i \to T_i'\right\}_{i=1}^n$ such that $Q(\omega, t_1, \ldots, t_n) = Q'(\omega, f_1(t_1), ..., f_n(t_n))$ for each ω and $(t_1, \ldots, t_n) \in \boldsymbol{T}$. In other words, two type spaces are equivalent if they formalize the same informational environment up to a relabeling of types.[1]

A distinct approach formalizes each agent's information as a partition of an expanded state space. Fix a measurable space X endowed with a non-atomic probability measure λ. A *signal* is a finite partition of $\Omega \times X$ whose elements belong to S, the set of non-empty measurable subsets of $\Omega \times X$. An element $s \in S$ is a *signal realization*. The interpretation is that a random variable drawn according to λ determines the signal realization conditional on the state. In particular, an agent with signal π_i observes the realization $s \in \pi_i$ that contains $(\omega, x) \in \Omega \times X$. Hence, the conditional probability of s given ω is $\lambda(s^\omega)$ where $s^\omega \equiv \{x \in X | (\omega, x) \in s\}$. The unconditional probability of s, denoted $p(s)$, is equal to $\sum_\omega \mu_0(\omega) \lambda(s^\omega)$. Let Π be the set of all signals. A *signal allocation* is a vector $\boldsymbol{\pi} = (\pi_1, ..., \pi_n)$ with $\pi_i \in \Pi$.

Every signal allocation is naturally associated with a type space in which the types are the signal realizations. Formally, signal allocation $\boldsymbol{\pi}$ *induces* the type space $(\pi_1 \times \cdots \times \pi_n, Q^{\boldsymbol{\pi}})$ with $Q^{\boldsymbol{\pi}}(\omega, s_1, \ldots, s_n) = \lambda((s_1 \cap \cdots \cap s_n)^\omega) \mu_0(\omega)$ for each ω and each $(s_1, ..., s_n) \in \boldsymbol{\pi}$. We use $\langle \boldsymbol{\pi} \rangle$ to denote the type space induced by $\boldsymbol{\pi}$. We say that a signal allocation $\boldsymbol{\pi}$ *represents* type space $(\boldsymbol{T}, Q)$ if $\langle \boldsymbol{\pi} \rangle \sim (\boldsymbol{T}, Q)$.

1.2 A benefit of signal formalism

The formalism of representing information sources as signals ("signal formalism" from hereon) has recently been used to study a number of topics in information economics. Gentzkow and Kamenica (2017) study environments where multiple senders simultaneously aim to persuade a receiver. Frankel and Kamenica (2019) derive ways to measure the amount of information generated by a piece of news. Brooks et al. (2022) investigate the distinctions between various notions of being "more informed." Brooks et al. (2024) study how to compare information sources in ways that are robust to potential presence of additional information.

These papers all draw on a particular benefit of signal formalism, namely a notational system that allows one to describe an agent's information without a reference to the information of other agents.[2] By contrast, type spaces require that the set of

[1] In formulating games of incomplete information, many treatments specify that each player's utility depends only on the vector of actions and the vector of types, without an explicit state of the world (Harsanyi 1967; Fudenberg and Tirole 1991). For our purposes, however, it is necessary to keep the state explicit as the fundamental source of uncertainty about which the agents have private information, as in Mertens and Zamir (1985).

[2] A related benefit is that signal formalism allows us to intepret a "piece of data" without a reference to who observes it. In particular, the informational content of a signal realization $s \in S$ is defined independently

agents and their information are all specified at the outset. To illustrate this difference, suppose there are two agents, Ann and Bob. If we use type space formalism and are initially unaware of Bob's existence, we describe Ann's information as some (T_A, Q_A). If we are unaware of Ann's existence, we describe Bob's information as some (T_B, Q_B). Upon learning that both Ann and Bob need to be considered, we have to describe a new, expanded type space $(T_A \times T_B, Q)$. By contrast, if we use signal formalism, we specify Ann's information as signal π_A (whether Bob exists or not). Similarly, we specify Bob's information as signal π_B. Upon learning that both Ann and Bob need to be considered, nothing new needs to be done: our description of Ann and Bob's information via signals already encodes the correlation structure between their information.

1.3 Two interpretations

The aforementioned benefit of signal formalism might seem like a mathematical sleight of hand; is it really possible to specify at the outset how one agent's information correlates with *every* other information source imaginable? The views on this may vary depending on the intepretation of signal formalism. As emphasized in Brooks et al. (2024), there are (at least) two distinct intepretations.

One intepretation is that $\Omega \times X$ is the "true" state space, in the tradition of Aumann (1976), that captures all uncertainty in the world, including the uncertainty that governs the realization of each agent's information.[3] Returning to the example above, under this interpretation, there is a unique signal that describes Ann's information. Thus, we can specify Ann's information whether Bob exists or not and similarly for Bob.

A different interpretation is that X is not some "true" dimension of the state space, but rather a modeling device that encodes the uncertainty conditional on the state. Under this interpretation, the use of signal formalism may require some caution. This note addresses two potential concerns that arise, which we describe next.

1.4 Our results

First, once we fix X and λ, the set of type spaces that are induced by signal allocations is clearly a strict subset of all possible type spaces. Is there any loss of generality in restricting attention to such type spaces? In other words, is every type space represented by some signal allocation? Theorem 1 shows that, under the maintained assumption that λ is a non-atomic measure on X, the answer is affirmative: every type space is represented by some signal allocation.

Second, one may worry that the appropriate choice of how to encode an agent's information as a signal may depend on what information other agents have. Returning to our example of Ann and Bob, suppose we initially consider Ann only, knowing that her information about the state is represented by (T_A, Q_A). There is a signal π_A that

of who the observer is and what alternative data they could have seen. Frankel and Kamenica (2019) draw heavily on this feature of signal formalism.

[3] The distinction between Ω and X is then typically the distinction between payoff-relevant and payoff-irrelevant dimensions of the state space.

represents (T_A, Q_A). In fact, there are many such signals.[4] Suppose we select some particular signal, say $\hat{\pi}_A$, to represent (T_A, Q_A). Then, Bob comes into the picture. We now wish to encode the type space $(T_A \times T_B, Q)$ as some signal allocation. Might we have to revise our previous choice for $\hat{\pi}_A$? Theorem 1 tells us there is a signal allocation (π_A, π_B) that represents $(T_A \times T_B, Q)$, but this does not imply that there is a π_B such that $\left(\hat{\pi}_A, \pi_B\right)$ represents $(T_A \times T_B, Q)$. If there were no such π_B, that would contradict our claim that the signal formalism allows us to represent an agent's beliefs and higher order beliefs, without reference to the information of others. Theorem 2, however, establishes that this is not a concern. There is in fact always a $\left(\hat{\pi}_A, \pi_B\right)$ that represents $(T_A \times T_B, Q)$. More generally, Theorem 2 establishes that the encoding of type spaces into signal allocations can be done "myopically", without consideration of other potential agents and their information.

1.5 Related literature

When $n = 1$, a type space corresponds to an experiment (Blackwell 1951). Green and Stokey (1978; 2022) consider the relationship between the Blackwell order on experiments and the refinement order on signals. In particular, they establish[5] that if experiment (T_1, Q) Blackwell dominates $\left(T_1', Q'\right)$, there exist X and λ and signal allocations (π_1) and $\left(\pi_1'\right)$ such that $\langle(\pi_1)\rangle \sim (T_1, Q)$, $\langle\left(\pi_1'\right)\rangle \sim \left(T_1', Q'\right)$ and π_1 refines π_1'.

Green and Stokey (1978; 2022) allow the choice of X and λ to depend on (T_1, Q) and $\left(T_1', Q'\right)$. Gentzkow and Kamenica (2017) establish a slightly stronger version of the result by pre-specifying X to be the unit interval $[0, 1]$ and λ to be the uniform distribution. While this strengthening is mathematically trivial, pre-specifying (X, λ) is important for applications since it is required in order for signal formalism to yield the notational benefits mentioned above. The particular choice of the unit interval and the uniform distribution provides further expositional benefit because the conditional probability of s given ω becomes simply the Lebesgue measure of s^{ω}. Indeed, all of the aforementioned papers that use signal formalism use the $[0, 1]$-uniform version. In this note, we fix (X, λ) at the outset but only impose the essential requirement of non-atomicity.

In Brooks et al. (2022), we consider whether the Green-Stokey result extends beyond pairs of experiments. We establish that, in general, the answer is negative. We construct a collection of experiments $\{(T_i, Q_i)\}_i$ such that there is no collection of signals $\{\pi_i\}_i$ with the property that $\langle(\pi_i)\rangle \sim (T_i, Q_i)$ for each i, and if (T_i, Q_i) Blackwell dominates $\left(T_j, Q_j\right)$ then π_i refines π_j for each i and j. However, we also show that if the collection $\{(T_i, Q_i)\}_i$ satisfies a certain property—if we consider

[4] Recall that this was not the case under the "true" state space interpretation of signal formalism. When X is just a modeling device to encode uncertainty, however, there is freedom in the choice of the encoding.

[5] Green and Stokey's paper, while formally published in 2022, has been available as a working paper at least since 1978. In the note accompanying the published paper, the authors write "when we tried to publish our paper in the late 1970s, a referee at the *Annals of Statistics* pointed out that the result in Theorem 1 had appeared as Proposition 13 (p. 1439) in... (Le Cam 1964). We are pleased that there is still enough interest in the issue to warrant publishing our paper now, which will perhaps make the result more accessible to economists."

the undirected graph whose nodes are the experiments and whose edges denote their Blackwell comparability, this graph has no cycles—then in fact there is always a collection of signals $\{\pi_i\}_i$ that induce the corresponding experiments, and such that if (T_i, Q_i) Blackwell dominates $\left(T_j, Q_j\right)$ then π_i refines π_j. The construction of these signals, however, cannot be done "myopically"; to build $\{\pi_i\}_i$, we might assign a provisional π_i to represent some (T_i, Q_i) but once we encounter a (T_k, Q_k) with $k > i$, we may have to go back and re-assign a different signal to represent (T_i, Q_i).[6] This provides a useful contrast to our Theorem 2.

2 Main results

The definitions we already provided suffice for the statement of our first result.

Theorem 1 *Given any type space, there exists a signal allocation that represents it.*

The proof (and the intuition) behind Theorem 1 are easiest to understand as consequences of our second result.

Toward the statement of that result, we say that the type space $\left(T'_1 \times \cdots \times T'_{n'}, Q'\right)$ *extends* the type space $(T_1 \times \cdots \times T_n, Q)$ if $n' \geq n$, $T'_i = T_i$ for all $i \leq n$, and Q is the marginal of Q' on $\Omega \times T_1 \times \cdots \times T_n$.

Theorem 2 *Suppose that the signal allocation $(\pi_1, \ldots, \pi_n)$ represents the type space $(T_1 \times \cdots \times T_n, Q)$. Suppose that $\left(T'_1 \times \cdots \times T'_{n'}, Q'\right)$ extends $(T_1 \times \cdots \times T_n, Q)$. There exists $(\pi_{n+1}, \ldots, \pi_{n'})$ such that the signal allocation $(\pi_1, \ldots, \pi_n, \pi_{n+1}, \ldots, \pi_{n'})$ represents $\left(T'_1 \times \cdots \times T'_{n'}, Q'\right)$.*

Proof of Theorem 2 We prove the theorem for the case where $n' = n + 1$. The general case follows by induction.

Since $\boldsymbol{\pi} \equiv (\pi_1, \ldots, \pi_n)$ represents $(T_1 \times \cdots \times T_n, Q)$, there are bijections $\{f_i : T_i \to \pi_i\}_{i=1}^n$ such that $Q\left(\omega, t_1, \ldots, t_n\right) = Q^{\boldsymbol{\pi}}\left(\omega, f_1\left(t_1\right), ..., f_n\left(t_n\right)\right)$ for each ω and $(t_1, \ldots, t_n) \in \boldsymbol{T}$.

For each $t \in T_1 \times \cdots \times T_n$, let

$$s_t = f_1\left(t_1\right) \cap \cdots \cap f_n\left(t_n\right).$$

Note that $\lambda(s_t^\omega)\mu_0\left(\omega\right) = Q^{\boldsymbol{\pi}}\left(\omega, f_1\left(t_1\right), \ldots, f_n\left(t_n\right)\right) = Q(\omega, t)$.

We partition each s_t^ω into subsets $\left(s_{t,t_{n+1}}^\omega\right)_{t_{n+1} \in T_{n+1}}$, so that $\lambda\left(s_{t,t_{n+1}}^\omega\right)\mu_0\left(\omega\right) = Q'\left(\omega, t, t_{n+1}\right)$. This is possible because λ is non-atomic[7] and

$$\sum_{t_{n+1} \in T_{n+1}} Q'(\omega, t, t_{n+1}) = Q(\omega, t) = \lambda\left(s_t^\omega\right)\mu_0\left(\omega\right).$$

[6] To see why, note that the signal π_i has to be specified with finitely many realizations, and any π_j with more realizations than π_i cannot be a coarsening of π_i.

[7] For our purposes, the key implication of non-atomicity is the *Darboux property*: For any $X' \subseteq X$ such that $\lambda\left(X'\right) > 0$ and for every real number $\alpha \in [0, 1]$, there is a $X'' \subseteq X'$ such that $\lambda\left(X''\right) = \alpha\lambda\left(X'\right)$.

For each $t_{n+1} \in T_{n+1}$, let

$$f_{n+1}(t_{n+1}) = \cup_{\omega \in \Omega, t \in T_1 \times \cdots \times T_n} s^{\omega}_{t,t_{n+1}}$$

and set

$$\pi_{n+1} = \{f_{n+1}(t_{n+1}) | t_{n+1} \in T_{n+1}\}.$$

Note that the $f_{n+1}(t_{n+1})$ sets are disjoint and clearly their union is all of $\Omega \times X$, so π_{n+1} is a signal. Moreover, f_{n+1} is clearly a bijection from T_{n+1} to π_{n+1}. Finally, by construction, for each ω and $(t, t_{n+1}) \in (T_1 \times \cdots T_n) \times T_{n+1}$, we have that $\left(\cap_{i=1}^{n+1} f_i(t_i)\right)^{\omega} = s^{\omega}_{t,t_{n+1}}$, and so

$$\lambda\left(\left(\cap_{i=1}^{n+1} f_i(t_i)\right)^{\omega}\right)\mu_0(\omega) = \lambda\left(s^{\omega}_{t,t_{n+1}}\right)\mu_0(\omega) = Q'(\omega, t, t_{n+1}),$$

Hence, $(\pi_i)_{i=1}^{n+1}$ represents $\left(T_1 \times \cdots \times T_{n+1}, Q'\right)$. □

With Theorem 2 in hand, it is easy to prove Theorem 1.

Proof of Theorem 1 Given some type space $(T_1 \times \cdots \times T_n, Q)$, we wish to construct a signal allocation that represents it. First consider the null type space (T_0, Q_0) where T_0 is a singleton. This type space is represented by the null signal allocation consisting of only the signal $\pi_0 = \{\Omega \times X\}$. Moreover, $\left(T_0 \times T_1 \times \ldots \times T_n, Q'\right)$ with $Q'(\omega, t_0, \ldots, t_n) = Q(\omega, t_1, \ldots, t_n)$ extends (T_0, Q_0). Hence, by Theorem 2, there is a signal allocation $(\pi_0, \pi_1, \ldots, \pi_n)$ that represents $\left(T_0 \times T_1 \times \cdots \times T_n, Q'\right)$. Consequently, $(\pi_1, \ldots, \pi_n)$ represents $(T_1, \ldots, T_n, Q)$. □

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